# ON THE CLASSIFICATION OF DEFECTIVE THREEFOLDS

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ABSTRACT. In this paper we give the full classification of irreducible projective threefolds whose $k$-secant variety has dimension smaller than the expected, for some $k \geq 2$ (see theorem 0.1 below). As pointed out in the introduction, the case $k = 1$ was already known before.

## INTRODUCTION

An irreducible, non–degenerate, projective variety $X$ of dimension $n$ in $\mathbb{P}^r$ is called $k$–*defective* if its $k$–secant variety $S^k(X)$ has dimension $s^{(k)}(X)$ smaller than the expected $\sigma^{(k)}(X)$, which is the minimum between $r$ and $n(k+1)+k$. The difference $\delta_k(X) = \sigma^{(k)}(X) - s^{(k)}(X)$ is called the $k$–*defect* of $X$.

The classification of defective varieties is important in the study of projective geometry and its applications. The subject goes back to several classical authors, like Terracini ([29]), Palatini ([25]) and Scorza ([26]), to mention a few. More recently the interest on defective varieties has been renewed by Zak's spectacular results on the classification of some important classes of smooth varieties (e.g. Severi varieties, Scorza varieties etc., [33]).

Beside the intrinsic interest of the subject, it turns out that the understanding of defective varieties is relevant also in other fields of Mathematics: expressions of polynomials as sums of powers and Waring type problems, polynomial interpolation, rank tensor computations and canonical forms, Bayesian networks, algebraic statistics etc. (see [9] as a general reference, [5], [18], [21]). Se also [28] and [32] as further examples of results on the subject which have nice applications to number theoretic problems.

In the present paper, we give the complete classification of complex defective threefolds.

One knows that curves are never defective (see [10]). Defective surfaces have been classified by Terracini in [30]. Terracini's result has been revisited and extended in [11], [4] and [6]. The case of 1–defective threefolds goes back to Scorza [26] and has been revisited in [7]. The case of smooth 1–defective threefolds was also examined by Fujita [16] and Fujita–Roberts [17]. Here we deal with the case of $k$–defective threefolds, for any $k > 1$, and we classify *minimally* $k$–defective threefolds, i.e. $k$–defective threefolds that are not $h$–defective for any $h < k$. This is of course sufficient to describe all defective threefolds.

The classification is obtained by using the classical tool of tangential projections. The basic invariants are the *tangential contact loci* (see §1). An important tool is also provided by Castelnuovo's theory on the growth of Hilbert functions (see

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[6], §6). After having recalled several basic notation, definitions and results in §1, we prove some generalities on defective threefolds in §2 and on the contact loci in §3. Then we prove our classification results in §§4, 5, 6. Indeed §4 is devoted to the analysis of the case in which the appropriate tangential contact locus is an irreducible divisor, §5 to the case in which the contact locus is reducible, §6 to the case in which the contact locus is an irreducible curve.

Now we state our classification result, obtained by summarizing Theorems 4.2, 4.5, 6.7, 6.7 and various corollaries and remarks accompanying them (for the definition of the invariant $n_k(X)$ see 1.10 below):

THEOREM 0.1. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, minimally $k$–defective threefold with $k \geq 2$. Then $X$ is in the following list:*

(1) *$X$ is contained in a cone over the 2–uple embedding of a threefold $Y$ of minimal degree $k - 1$ in $\mathbb{P}^{k+1}$, with vertex either a point, hence $r = 4k + 2$ and $\delta_k(X) = 1$, $s^{(k)}(X) = r - 1 = 4k + 1$, $n_k(X) = 1$, or a line, hence $r = 4k + 3$ and $\delta_k(X) = 1$, $s^{(k)}(X) = r - 1 = 4k + 2$, $n_k(X) = 2$ (see Example 4.3, (1));*

(2) *$k = 3$ and either $r = 4k + 2 = 14$, $s^{(k)}(X) = r - 1 = 13$, $n_k(X) = 1$, and $X$ is the 2–uple embedding of a hypersurface $Y$ in $\mathbb{P}^4$ with $\deg(Y) \geq 3$ or $r = 4k + 3 = 15$, $s^{(k)}(X) = r - 1 = 14$, $n_k(X) = 2$, and $X$ is contained in the cone with vertex a point over the 2–uple embedding of a hypersurface $Y$ as above (see Example 4.3, (2));*

(3) *either $r = 4k + 2$, $\delta_k(X) = 2$, $n_k(X) = 1$, $s^{(k)}(Y') = r - 1 = 4k + 1$, and $X$ is the 2–uple embedding of a threefold $Y$ of degree $k$ in $\mathbb{P}^{k+1}$ with curve sections of arithmetic genus 1 or $r = 4k + 3$, $\delta_k(X) = 1$, $n_k(X) = 2$, $s^{(k)}(Y') = r - 1 = 4k + 2$, and $X$ is contained in the cone with vertex a point over the 2–uple embedding of a threefold $Y$ as above (see Example 4.3, (3));*

(4) *$r = 4k + 3$, $\delta_k(X) = 1$, $n_k = 2$ and $s^{(k)}(X) = r - 1 = 4k + 2$, and $X$ is the 2–uple embedding of a threefold $Y$ of degree $k$ in $\mathbb{P}^{k+1}$ with curve sections of genus 0, which is either a cone with vertex a line over a smooth rational curve of degree $k$ in $\mathbb{P}^{k-1}$ or it has a double line (see Example 4.3, (4));*

(5) *$k = 4$, $r = 4k + 3 = 19$, $\delta_4(X) = 1$, $n_4 = 2$, $s^{(k)}(X) = r - 1 = 18$, and $X$ is the 2–uple embedding of a threefold $Y$ in $\mathbb{P}^5$ with $\deg(Y) \geq 5$, contained in a quadric (see Example 4.3, (5));*

(6) *$k \geq 4$, $r = 4k + 3$, $\delta_k(X) = 1$, $n_k = 2$, $s^{(k)}(X) = r - 1 = 4k + 2$, and $X$ is the 2–uple embedding of a threefold $Y$ of degree $k + 1$ in $\mathbb{P}^{k+1}$ with curve sections of arithmetic genus 2 (see Example 4.3, (6));*

(7) *$r = 4k + 3 - i$, $i = 0, 1$, $\delta_k(X) = 1$, $s^{(k)}(X) = r - 1 = 4k + 3 - i$, $n_k = 2 - i$, and $X$ is contained in a cone with vertex a space of dimension $k - i$ over the 2–uple embedding of a surface $Y$ of minimal degree $k$ in $\mathbb{P}^{k+1}$ (see Theorem 4.5, case (1));*

(8) *$k = 2$, $r = 4k + 3 = 11$, $\delta_k(X) = 1$, $s^{(k)}(X) = r - 1 = 10$, $n_k = 2$, and $X$ is contained in a cone with vertex a line over the 2–uple embedding of a surface $Y$ of $\mathbb{P}^3$ with $\deg(Y) \geq 3$ (see Theorem 4.5, case (2));*

(9) $k \geq 3$, $r = 4k + 3$, $\delta_k(X) = 1$, $s^{(k)}(X) = r - 1 = 4k + 2$, $n_k = 2$, and $X$ sits in a cone with vertex of dimension $k - 1$ over the 2–uple embedding of a surface $Y$ of degree $k + 1$ in $\mathbb{P}^{k+1}$ with curve sections of arithmetic genus 1 (see Theorem 4.5, case (3));

(10) $r \geq 4k + 3$, $\delta_k = 1$, $s^{(k)}(X) = 4k + 2$, $n_k = 2$ and $X$ is contained in a cone with vertex of dimension $k - 1$, and not smaller, over a surface which is not $k$–weakly defective (see Proposition 5.2);

(11) $r \geq 4k+3$, $\delta_k(X) = 1$, $s^{(k)}(X) = 4k+2$, $n_k = 2$ and $X$ is contained in a cone with vertex of dimension $2k$, and not smaller, over a curve (see Proposition 5.3);

(12) $r \geq 4k + 2$, $s^{(k)}(X) = 4k + 1$, $n_k = 1$ (hence $\delta_k(X) = 1$, if $r = 4k + 2$, whereas $\delta_k(X) = 2$, if $r > 4k + 2$) and $X$ is contained in a cone with vertex of dimension $2k - 1$, and not smaller, over a curve (see Proposition 5.4);

(13) $4k + 3 \leq r \leq 4k + 5$, $s^{(k)}(X) = 4k + 2$, $\delta_k(X) = 1$, $n_k = 1$ and $X$ is either the 2–uple embedding of a threefold of minimal degree $k$ in $\mathbb{P}^{k+2}$ (hence $r = 4k + 5$), or the projection from a point of $\mathbb{P}^{4k+5}$ of the 2–uple embedding of a threefold $Y$ of minimal degree $k$ in $\mathbb{P}^{k+2}$, or the projection from a line $\ell \subset \mathbb{P}^{4k+5}$ of the 2–uple embedding $Y' \subset \mathbb{P}^{4k+5}$ of a threefold $Y$ of minimal degree $k$ in $\mathbb{P}^{k+2}$ (see Example 6.2);

(14) $r = 4k + 3$, $s^{(k)}(X) = r - 1 = 4k + 2$, $\delta_k(X) = 1$, $n_k(X) = 2$ and $X$ is linearly normal, contained in the intersection of a space of dimension $4k + 3$ with the Segre embedding of $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$ in $\mathbb{P}^{k^2+4k+3}$, but not lying in the 2–uple embedding of $\mathbb{P}^{k+1}$, and such that the two projections of $X$ to $\mathbb{P}^{k+1}$ span $\mathbb{P}^{k+1}$ (see Coorollary 6.9).

Cases (1)–(9) correspond to the situation in which the tangential contact locus is an irreducible surface, case (10) corresponds to the situation in which the tangential contact locus is a reducible curve (a union of lines), cases (11)–(12) correspond to the situation in which the tangential contact locus is a reducible surface, cases (13)–(14) correspond to the situation in which the tangential contact locus is an irreducible curve, namely a rational normal curve of degree $2k$.

All threefolds in this list are actually minimally $k$–defective. Threefolds of types (1)–(9) and (14) are not $h$–defective for any $h > k$. The same is true for threfolds of type (13) with $r < 4k + 5$. Threefolds of type (13) with $r = 4k + 5$ are also $(k + 1)$–defective. Threefolds of types (10)–(12) can be $h$–defective for $h > k$ (see Remark 5.5).

Brief discussions about the existence of smooth defective threefolds are contained in Example 4.3 and Remarks 4.6 and 5.5 below.

# 1. PRELIMINARIES AND NOTATION

1.1.   In this paper we work over the complex field $\mathbb{C}$. Let $X \subseteq \mathbb{P}^r$ be an irreducible projective scheme over $\mathbb{C}$. We will denote by $\deg(X)$ the *degree* of $X$ and by $\dim(X)$ the *dimension* of $X$. If $X$ is reducible, by $\dim(X)$ we mean the maximum of the

dimensions of its irreducible components. We will denote by $\mathcal{O}_X$ the structure sheaf of $X$ and by $\mathcal{I}_X$ the ideal sheaf of $X$ in $\mathbb{P}^r$.

If $X$ is irreducible, by a *general point* of $X$ we mean a point which can vary in some dense open Zariski subset of $X$.

One says that $X$ is *rationally connected* is for $x, y \in X$ general points, there is a rational curve in $X$ containing them.

If $Y \subset \mathbb{P}^r$ is a subset, we denote by $< Y >$ the span of $Y$. We will say that $Y$ is *non–degenerate* if $< Y >= \mathbb{P}^r$.

1.2.  If $X \subseteq \mathbb{P}^r$ is an irreducible, projective, non degenerate variety, we will say that it is a *cone* over a variety $Y$, if there is a projective subspace $V$ of dimension $v$, called a *vertex* of $X$, and a projective subspace $W$ of dimension $r - v - 1$, containing $Y$, such that $X = \cup_{x \in X, y \in V} < x, y >$.

Let $V$ be a projective subspace of $\mathbb{P}^r$ of dimension $v$, not containing $X$. We will say that $X$ *sits in the cone over a variety* $Y \subset \mathbb{P}^{r-v-1}$ *with vertex* $V$ if and only if the image of the projection of $X$ from $V$ is $Y$.

1.3.  Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate projective variety. One has the following famous bound:

$$(1) \qquad\qquad \deg(X) \geq r - \dim(X) + 1$$

which is sharp: the varieties achieving the bound (1) are called *varieties of minimal degree* and their classification is well known (see [15]): curves of minimal degree are the rational normal curves, surfaces of minimal degree are either rational normal scrolls or the Veronese surface in $\mathbb{P}^5$, etc.

We will denote by $h_X$ the *Hilbert function* of $X$, namely for any non–negative integer $k$, $h_X(k)$ is the dimension of the image of the restriction map:

$$\rho_{X,k} : H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \to H^0(X, \mathcal{O}_X(k)).$$

Recall that one says that $X$ is $k$–*normal* if the map $\rho_{X,k}$ is surjective, i.e. if $H^1(\mathbb{P}^r, \mathcal{I}_X(k)) = 0$. One says *linearly normal, quadratically normal* etc. instead of 1-normal, 2–normal, etc. Then $X$ is linearly normal if and only if the hyperplane system $\mathcal{H}$ is complete. The variety $X$ is said to be *projectively normal* if it is $k$–normal for every $k \geq 1$.

We will need the following result of Castelnuovo's theory from [6] (see Theorem 6.1):

THEOREM 1.1. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective variety of dimension $n$ and degree $d$. Set $\iota := \iota(X) = \min\{d + n - r - 1, r - n\}$. Then:*

$$(2) \qquad\qquad h_X(2) \geq \iota + r(n+1) - \frac{n(n-1)}{2} + 1.$$

*If $d \leq 2(r - n) + 1$ then the following are equivalent:*
  (i) *the equality holds in (2);*
  (ii) *the general curve section of $X$ is linearly normal of genus $\iota$;*

(iii) *the general curve section of $X$ is projectively normal of genus $\iota$.*

Remark that, if $X$ is non–degenerate, the invariant $\iota(X)$ defined in the previous statement is non–negative because of (1).

Let us now mention the following interesting proposition which follow from results of [3], Corollary 3.3:

PROPOSITION 1.2. *Let $X \subset \mathbb{P}^r$ be a smooth isomorphic projection of a variety of minimal degree in $\mathbb{P}^{r+1}$. Then $X$ is $k$–normal for every $k \geq 2$.*

Using it, we can prove the following proposition, which will be useful later:

LEMMA 1.3. *Let $Y \subset \mathbb{P}^r$ be a non–degenerate threefold of minimal degree $r - 2$ and let $Y' \subset \mathbb{P}^{r-1}$ be a projection of $Y$ from a point $p \notin Y$. Assume that $Y'$ is non–singular in codimension 1. Then $h_{Y'}(2) \geq 4r - 4$ and the equality holds if and only if one of the following cases occurs:*

(i) *$Y'$ is a cone with vertex a line over a smooth rational curve $C$ of degree $r - 2$ in $\mathbb{P}^{r-3}$;*

(ii) *$Y'$ has a line $L$ of double points, and the pull back of $L$ on $Y$ is a conic whose plane contains the centre of projection $p$.*

*Proof.* We let $S \subset \mathbb{P}^{r-2}$ be the general surface section of $Y'$ and $C \subset \mathbb{P}^{r-3}$ the general curve section of $S$. Notice that $C$ is a smooth rational curve of degree $r - 2$. Standard diagram chasing gives:

$$h_{Y'}(2) - h_{Y'}(1) = h_S(2) + \dim(\ker\{H^1(\mathbb{P}^{r-1}, \mathcal{I}_{Y'}(1)) \to H^1(\mathbb{P}^{r-1}, \mathcal{I}_{Y'}(2))\})$$

$$h_S(2) - h_S(1) = h_C(2) + \dim(\ker\{H^1(\mathbb{P}^{r-2}, \mathcal{I}_S(1)) \to H^1(\mathbb{P}^{r-2}, \mathcal{I}_S(2))\})$$

(see [8], p. 30). Now:

$$h_{Y'}(1) = r, \quad h_S(1) = r - 1$$

and

$$h_C(2) = 2r - 3$$

by Proposition 1.2. Thus:

$$h_{Y'}(2) = 4r - 4 +$$

(3)            $$+ \dim(\ker\{\mathbb{P}^{r-1}, H^1(\mathcal{I}_{Y'}(1)) \to H^1(\mathbb{P}^{r-1}, \mathcal{I}_{Y'}(2))\}) +$$

$$+ \dim(\ker\{H^1(\mathbb{P}^{r-2}, \mathcal{I}_S(1)) \to H^1(\mathbb{P}^{r-2}, \mathcal{I}_S(2))\}).$$

This proves the first assertion. Moreover $h_{Y'}(2) = 4r - 4$ yields:

$$\dim(\ker\{H^1(\mathbb{P}^{r-1}, \mathcal{I}_{Y'}(1)) \to H^1(\mathbb{P}^{r-1}, \mathcal{I}_{Y'}(2))\}) = 0$$

$$\dim(\ker\{\mathbb{P}^{r-2}, H^1(\mathcal{I}_S(1)) \to H^1(\mathbb{P}^{r-2}, \mathcal{I}_S(2))\}) = 0.$$

This implies that both $Y'$ and $S$ are singular. Suppose in fact that $Y'$ is smooth. Then $h^1(\mathbb{P}^{r-1}, \mathcal{I}_{Y'}(1)) = 1$ and $h^1(\mathbb{P}^{r-1}, \mathcal{I}_{Y'}(2)) = 0$ by Proposition 1.2. The same argument works for $S$.

Let $x \in S$ be a singular point. If $S$ is a cone with vertex $x$ we are in case (i). Suppose $S$ is not a cone and let $\mu$ be the multiplicity of $S$ at $x$. The projection of $S$ from $x$ is a non–degenerate surface of degree $r - 2 - \mu$ in $\mathbb{P}^{r-3}$. This proves that $\mu = 2$. We claim that $x$ is the only singular point of $S$. In fact $S$ is the projection of a surface $T \subset \mathbb{P}^{r-1}$ of minimal degree from a point $p \notin T$. Since $S$ is not a cone, then $T$ is not a cone, hence it is smooth (see [15]). The singular point $x$ of $S$ arises from a secant (or tangent) line $\ell$ to $T$ passing through $p$. Suppose there is another singular point $y \in S$. This would correspond to some other secant line $\ell'$ to $T$ containing $p$. The plane $\Pi = <\ell, \ell'>$ would then be 4–secant to $T$, and this is possibe only if $\Pi$ intersect $T$ along a conic (see again [15]). This would in turn yield a double line for $S$ and codimension 1 singularities for $Y$, a contradiction. In conclusion we are in case (ii).

Conversely, if we are in case (i), one has

$$h^0(\mathbb{P}^{r-1}, \mathcal{I}_{Y'}(2)) = h^0(\mathbb{P}^{r-3}, \mathcal{I}_C(2)) = \binom{r-1}{2} - 2r + 3$$

(see again Proposition 1.2). Hence:

$$h_{Y'}(2) = \binom{r+1}{2} - \binom{r-1}{2} + 2r - 3 = 4r - 4.$$

If we are in case (ii), we have $h^1(\mathbb{P}^{r-1}, \mathcal{I}_{Y'}(1)) = h^1(\mathbb{P}^{r-2}, \mathcal{I}_S(1)) = 0$, because the singular varieties $Y'$ and $S$ are linearly normal, and formula (3) yields $h_{Y'}(2) = 4r - 4$.                                                                        ∎

It is clear that one can prove a similar more general result for projections of varieties of minimal degree of any dimensions. Since we will not need it, we do not dwell on this here.

We also record the following:

PROPOSITION 1.4. *Let $Y \subset \mathbb{P}^{k+1}$ be an irreducible, non–degenerate, threefold of degree $k$ with smooth curve sections of genus $0$ and a singular line. Then:*

(i) *either $Y$ is a cone over a smooth rational curve of degree $k$ in $\mathbb{P}^{k-1}$,*
(ii) *or $Y$ is the projection in $\mathbb{P}^{k+1}$ of a threefold $Z$ of minimal degree $k$ in $\mathbb{P}^{k+2}$ from a point $p \notin Z$, containing a conic sitting in a plane passing through $p$.*

*Proof.* One knows that the threefold $Y$ is the projection in $\mathbb{P}^{k+1}$ of a threefold $Z$ of minimal degree $k$ in $\mathbb{P}^{k+2}$ from a point $p \notin Z$. If $Z$ is a cone with vertex a line, we are in case (i). If $Z$ is a cone with vertex a point, then $Y$ is also a cone. However the point $p$ has to sit on a secant line $\ell$ to $Z$ and we are in case (ii): the conic here is reducible in the two generators of the cone $Z$ passing through the intersection

points of $\ell$ with $Z$. If $Z$ is smooth, then again $p$ has to sit on a secant line $\ell$ to $Z$. The argument to show that we are in case (ii) then goes as in the proof of Lemma 1.3. ∎

1.4. Let $X$ be an irreducible, projective variety. We will use the symbol $\equiv$ to denote linear equivalence of Weil divisors, or linear systems, on $X$. If $D$ is a divisor, we will denote, as usual, by $|D|$ the complete linear system of $D$.

Let $H$ be a hyperplane divisor on $X$. We will denote by $\mathcal{H} \subset |H|$ the (possibly not complete) *hyperplane system*, i.e. the linear system cut out on $X$ by the hyperplanes of $\mathbb{P}^r$. If we assume that $X$ is non–degenerate, then $\mathcal{H}$ has dimension $r$. In this case, by abusing notation, we will sometimes identify the divisor $H$ with the unique hyperplane which cuts $H$ on $X$.

If $\mathcal{L}$ is a linear system of dimension $r$ of Weil divisors on $X$, we will denote by:

$$\phi_{\mathcal{L}} : X \dashrightarrow \mathbb{P}^r$$

the rational map defined by $\mathcal{L}$.

If $p_1, ..., p_k \in X$ are smooth points and $m_1, ..., m_k$ are positive integers, we will denote by $\mathcal{L}(-m_1 p_1 - ... - m_k p_k)$ the sublinear system of $\mathcal{L}$ formed by all divisors in $\mathcal{L}$ with multiplicity at least $m_i$ at $p_i$, $i = 1, ..., k$.

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are linear systems of Weil divisors on a $X$, we define $\mathcal{L}_1 + \mathcal{L}_2$ as the minimal linear system of Weil divisors on $X$ containing all divisors of the form $D_1 + D_2$ where $D_i \in \mathcal{L}_i$, $i = 1, 2$. The linear system $\mathcal{L}_1 + \mathcal{L}_2$ is called the *minimal sum* of $\mathcal{L}_1$ and $\mathcal{L}_2$.

If $\mathcal{L}$ is a linear system on $X$, one writes $2\mathcal{L}$ instead of $\mathcal{L} + \mathcal{L}$. Similarly one can consider the linear system $h\mathcal{L}$ for all positive integers $h$.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be linear systems of Weil divisors on $X$ of dimensions $r_1, r_2$. Set $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, and suppose $\mathcal{L}$ has dimension $r$. One can consider the maps:

$$\phi_{\mathcal{L}_i} : V \to \mathbb{P}^{r_i}, i = 1, 2$$

$$\phi_{\mathcal{L}} : V \to \mathbb{P}^r$$

determined by the linear systems in question. It is clear that:

$$\phi_{\mathcal{L}} = \psi \circ (\phi_{\mathcal{L}_1} \times \phi_{\mathcal{L}_2})$$

where:

$$\psi : \mathbb{P}^{r_1} \times \mathbb{P}^{r_2} \to \mathbb{P}^{r_1 r_2 + r_1 + r_2}$$

is the *Segre embedding*.

Similarly if $\mathcal{L}$ is a linear system on $V$ with dimension $r$, and

$$\phi_{\mathcal{L}} : V \to \mathbb{P}^r$$

is the corresponding map, then for any positive integer $h$ one has:

$$\phi_{h\mathcal{L}} = \psi_h \circ \phi_{\mathcal{L}}$$

where:

$$\psi_h : \mathbb{P}^r \to \mathbb{P}^{\binom{r+m}{m}-1}$$

is the *h-th Veronese embedding.*

We will need the following lemma:

LEMMA 1.5. *Let $C$ be a smooth, irreducible, projective curve of genus $g$ and let $\mathcal{L}_i$, $i = 1, 2$, be base point free linear systems on $C$. Set $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, $\dim(\mathcal{L}) = r$, $\dim(\mathcal{L}_i) = r_i$, $i = 1, 2$. One has $r \geq r_1 + r_2$. If $\mathcal{L}$ is birational and $r = r_1 + r_2$, then $g = 0$. If moreover $r_1 = r_2$, then $\mathcal{L}_1 = \mathcal{L}_2$ and this series is complete.*

*Proof.* The inequality $r \geq r_1 + r_2$ follows from Hopf's lemma (see [2], p. 108). If the equality holds, then $g = 0$ by [2], Exercise B-1, p. 137. Suppose now that, in addition, $r_1 = r_2$. Then by [2], Exercise B-4, p. 138, the two series are equal, and by the birationality assumption on $\mathcal{L}$, also $\mathcal{L}_1 = \mathcal{L}_2$ is birational. If the series is not complete, then its image is a non-normal rational curve $\Gamma$ in $\mathbb{P}^{r_1}$. Then, by Theorem 1.1, $h_\Gamma(2) \geq 2r_1 + 2$, a contradiction. $\blacksquare$

Let us now recall a definition:

DEFINITION 1.6. We will say that a linear system $\mathcal{L}$ on a smooth, projective variety $Z$ is *birational* [resp. *very big*] if the map $\phi_{\mathcal{L}}$ determined by $\mathcal{L}$ is birational [resp. generically finite] to its image. The system $\mathcal{L}$ is *big* if the complete system determined by some multiple of $\mathcal{L}$ is big, or, equivalently, if the complete system determined by some multiple of $\mathcal{L}$ is birational.

The following definition is standard:

DEFINITION 1.7. If $W$, $Z$ are varieties with $W \subset Z$ and $\mathcal{L}$ is a linear system of Cartier divisors on $Z$, one denotes by $\mathcal{L}_{|W}$ the *restriction* of $\mathcal{L}$ to $W$ defined as follows. The system $\mathcal{L}$ corresponds to a vector subspace $V$ of $H^0(Z, \mathcal{O}_Z(L))$, with $L$ a divisor in $\mathcal{L}$, and $\mathcal{L}_{|W}$ corresponds to the image of $V$ via the restriction map $H^0(Z, \mathcal{O}_Z(L)) \to H^0(W, \mathcal{O}_W(L))$.

The following lemma is immediate and the proof can be left to the reader:

LEMMA 1.8. *Let $Z$ be a smooth, projective variety, let $\mathcal{L}$ be a linear system on $Z$ and let $\mathcal{V}$ be a family of closed subvarieties of $Z$ such that a general element $V$ of $\mathcal{V}$ is smooth and irreducible. Suppose that if $z \in Z$ is a general point, there is a variety $V$ in $\mathcal{V}$ containing $z$. Then if $\mathcal{L}$ is birational [resp. very big, big] and $V$ is a general element of $\mathcal{V}$, then $\mathcal{L}_{|V}$ is also birational [resp. very big, big].*

1.5.  Let $X$ be an irreducible projective variety.  For any positive integer $h$ we let $\mathrm{Sym}^h(X)$ be the $k$–fold symmetric product of $X$.  If $p_1, ..., p_h$ are points in $X$, we denote by $[p_1, ..., p_h]$ the corresponding point in $\mathrm{Sym}^h(X)$.  Namely one has a surjective morphism:

$$\pi_{X,h} : (p_1, ..., p_h) \in X^h \to [p_1, ..., p_h] \in \mathrm{Sym}^h(X)$$

which is a finite covering of degree $h!$ and the monodromy, or Galois, group of this covering is the full symmetric group $\mathcal{S}_h$.

1.6.  Let $X$ be an irreducible, projective variety. Let $\mathcal{D} = \{D_y\}_{y \in Y}$ be an algebraic family of Weil divisors on $X$ parametrized by an projective variety $Y$. We will constantly assume that $\mathcal{D}$ is *effectively parametrized* by $Y$, i.e. that the corresponding map of $Y$ to the appropriate Hilbert scheme of divisors on $X$ is generically finite. One says that $\mathcal{D}$ is *irreducible of dimension $m$* if $Y$ is.

An irreducible algebraic family $\mathcal{D}$ of dimension $m$ on $X$ is called an *involution* if there is one single divisor of $\mathcal{D}$ coontaining $m$ general points of $X$.

Involutions have been studied in §5 of [6] to which we defer the reader for details. In particular in [6] a classical theorem of Castelnuovo and Humbert concerning involutions on curves (see [6], Proposition 5.9), has been extended to higher dimensional varieties (see [6], Proposition 5.10). For the reader's convenience we recall here the result we will need later on:

THEOREM 1.9. *Let $X$ be an irreducible, projective variety of dimension $n > 1$. Let $\mathcal{D}$ be an $m$–dimensional involution with no fixed divisors and such that its general element is reduced. Then either $\mathcal{D}$ is a linear system of Weil divisors or it is* composed with a pencil, *i.e. there is a map $f : X \dashrightarrow C$ of $X$ to an irreducible curve $C$ such that $\mathcal{D}$ is the pull-back, via $f$, of an involution on $C$. If $m > 1$, the general element of $\mathcal{D}$ is reducible if and only if $\mathcal{D}$ is composed with a pencil.*

1.7.  Let $X \subset \mathbb{P}^r$ an irreducible, non–degenerate projective variety of dimension $n$. Let $k$ be a non–negative integer and let $S^k(X)$ be the $k$–*secant variety* of $X$, i.e. the Zariski closure in $\mathbb{P}^r$ of the set:

$$\{x \in \mathbb{P}^r : x \text{ lies in the span of } k + 1 \text{ independent points of } X\}$$

Of course $S^0(X) = X$, $S^r(X) = \mathbb{P}^r$ and $S^k(X)$ is empty if $k \geq r + 1$. Moreover, for any $k \geq 0$, one has $S^k(X) \subseteq S^{k+1}(X)$. We will write $S(X)$ instead of $S^1(X)$ and we will assume $k \leq r$ from now on.

One can consider the *abstract $k$–th secant variety* $S_X^k$ of $X$, i.e. $S_X^k \subseteq \mathrm{Sym}^k(X) \times \mathbb{P}^r$ is the Zariski closure of the set of all pairs $([p_0, ..., p_k], x)$ such that $p_0, ..., p_k \in X$ are linearly independent points and $x \in < p_0, ..., p_k >$. One has the surjective map $p_X^k : S_X^k \to S^k(X) \subseteq \mathbb{P}^r$, i.e. the projection to the second factor. Hence:

$$(4) \qquad s^{(k)}(X) := \dim(S^k(X)) \leq \min\{r, \dim(S_X^k)\} = \min\{r, n(k + 1) + k\}$$

The right hand side of (4) is called the *expected dimension* of $S^k(X)$ and will be denoted by $\sigma^{(k)}(X)$. One says that $X$ is *k-defective* when strict inequality holds in (4). One says that:

$$\delta_k(X) := \sigma^{(k)}(X) - s^{(k)}(X)$$

is the *k–defect* of $X$. The variety $X$ is called *defective* if it is *k*–defective for some $k \geq 1$. Observe that, if $r \geq n(k+1) + k$, then $X$ is *k*–defective if and only if there are infinitely many $(k+1)$–secant $\mathbb{P}^k$'s passing through the general point of $S^k(X)$. Notice also that, if $X$ is $h$–defective for some $h \geq 1$, then it is $k$–defective for all $k$ such that $k \geq h$ and $s^{(k)} < r$. If $X$ is $k$–defective but not $(k-1)$–defective, then we will say that $X$ is *minimally k–defective*.

We will write $s^{(k)}, \sigma^{(k)}, \delta_k$ etc. instead of $s^{(k)}(X), \sigma^{(k)}(X), \delta_k(X)$, if there is no danger of confusion.

1.8.   If $p$ is a smooth point of $X$, we denote by $T_{X,p}$ the *tangent space* to $X$ at $p$. If $\Pi$ is a projective subspace of $\mathbb{P}^r$, we say that $\Pi$ is *tangent* to $X$ at $p$ if either:

- $\dim(\Pi) \leq \dim(X)$ and $p \in \Pi \subseteq T_{X,p}$, or
- $\dim(\Pi) \geq \dim(X)$ and $T_{X,p} \subseteq \Pi$.

Let $k$ be a positive integer and let $p_1, ..., p_k$ be points of $X$. We denote by $T_{X,p_1,...,p_k}$ the span of $T_{X,p_i}, i = 1, ..., k$.

If $X \subset \mathbb{P}^r$ is a projective variety, Terracini's Lemma describes the tangent space to $S^k(X)$ at a general point of it and gives interesting information in case $X$ is $k$–defective (see [29] or, for modern versions, [1], [6], [10], [33]). We may state it as follows:

THEOREM 1.10. *(Terracini's Lemma) Let $X \subset \mathbb{P}^r$ be an irreducible, projective variety. If $p_0, ..., p_k \in X$ are general points and $x \in < p_0, ..., p_k >$ is a general point, then:*

$$T_{S^k(X),x} = T_{X,p_0,...,p_k}.$$

*If $X$ is $k$–defective, then:*

(i) *$T_{X,p_0,...,p_k}$ is tangent to $X$ along a variety $\Gamma := \Gamma_{p_0,...,p_k}$ of positive dimension $\gamma_k := \gamma_k(X)$ containing $p_0, ..., p_k$;*

(ii) *the general hyperplane $H$ containing $T_{X,p_0,...,p_k}$ is tangent to $X$ along a variety $\Sigma := \Sigma(H) := \Sigma_{p_0,...,p_k}(H)$ of positive dimension $\epsilon_k := \epsilon_k(X)$ containing $p_0, ..., p_k$.*

*One has $\Gamma \subseteq \Sigma$ and therefore $\gamma_k \leq \epsilon_k$. Moreover one has:*

$$(5) \qquad\qquad k \leq \dim(< \Gamma >) \leq \dim(< \Sigma >) \leq k\epsilon_k + k + \epsilon_k - \delta_k.$$

We may and will assume that all irreducible components of $\Gamma$ and of $\Sigma$ contain some of the points $p_0, ..., p_k$. Otherwise we simply get rid of those components that do not do so. We will call $\Gamma$ the *tangential k–contact locus* of $X$ at $p_0, ..., p_k$. Similarly, for any hyperplane $H$ containing $T_{X,p_0,...,p_k}$, we will call $\Sigma := \Sigma(H)$ the

$k$–contact locus of $H$ at $p_0, ..., p_k$. We will call $\epsilon_k(X)$ the $k$-singular defect of $X$ and $\gamma_k(X)$ the $k$-tangential defect of $X$.

The following is a well known, straightforward application of Terracini's lemma (see [33]):

PROPOSITION 1.11. Let $X \subset \mathbb{P}^r$ be an irreducible, projective, non–degenerate variety. If for some $k \geq 0$ one has $s^{(k)}(X) = s^{(k+1)}(X)$, then $s^{(k)}(X) = r$. In particular, if $s^{(k)}(X) = r - 1$, then $s^{(k+1)}(X) = r$.

We record also the following result whose easy proof can be left to the reader:

PROPOSITION 1.12. Let $X \subset \mathbb{P}^r$ be an irreducible, projective, non–degenerate variety of dimension $n$. let $\Pi \subset \mathbb{P}^r$ be a projetive subspace of dimension $s$ and let $\pi$ be the projection of $\mathbb{P}^r$ from $\Pi$ to $\mathbb{P}^{r-s-1}$. Let $Y = \pi(X)$ and let $m$ be its dimension. Then:

(i) for every positive integer $k$ one has $\pi(S^k(X)) = S^k(Y)$, hence $s^{(k)}(Y) \leq s^{(k)}(X) \leq s^{(k)}(Y) + s + 1$;

(ii) if $n = m$ and $s^{(k)}(Y) = (k+1)n + k$ then also $s^{(k)}(X) = (k+1)n + k$;

(iii) if $n = m$, $s^{(k)}(Y) = s^{(k)}(X)$ and $Y$ is $k$–defective [resp. minimally $k$–defective] then also $X$ is $k$–defective [resp. minimally $k$–defective] and $\pi$ maps the tangential $k$–contact locus [resp. the $k$–contact locus] of $X$ to the tangential $k$–contact locus [resp. the $k$–contact locus] of $Y$.

1.9. We recall from [6] the definition of a $k$–weakly defective variety, i.e. a variety $X \subset \mathbb{P}^r$ such that if $p_0, ..., p_k \in X$ are general points, then the general hyperplane $H$ containing $T_{X, p_0, ..., p_k}$ is tangent to $X$ along a variety $\Sigma := \Sigma(H) := \Sigma_{p_0, ..., p_k}(H)$ of positive dimension $\epsilon_k := \epsilon_k(X)$ containing $p_0, ..., p_k$. Similarly we can say that a variety $X \subset \mathbb{P}^r$ is $k$–weakly tangentially defective if whenever $p_0, ..., p_k \in X$ are general points, then $T_{X, p_0, ..., p_k}$ is tangent to $X$ along a variety $\Gamma := \Gamma_{p_0, ..., p_k}$ of positive dimension $\gamma_k : \gamma_k(X)$ containing $p_0, ..., p_k$. Of course $k$–weakly tangentially defectiveness implies $k$–weakly defectiveness, but the converse does not hold in general. Moreover, by Terracini's lemma, a $k$–defective variety is also $k$–weakly defective but again the converse does not hold in general (see [6]).

REMARK 1.13. A curve which is not a line is never $k$–weakly (tangentially) defective for any $k$. Hence a curve is never $k$–defective.

For a variety $X$ being 0–tangentially defective means that it is *developable*, i.e. the Gauss map of $X$ has positive dimensional fibres, in particular, according to Zak's theorem on tangencies (see [33]), $X$ must be singular, unless $X$ is a linear space. Instead, 0–weakly defective means that the dual variety of $X$ is not a hypersurface. In the surface case this happens if and only if the surface is developable, i.e. if and only if the surface is either a cone or the tangent developable to a curve (see [19]). However this is no longer the case if $\dim(X) > 2$ (see [13]). Finally no variety is 0–defective.

1.10. Let $X \subset \mathbb{P}^r$ be, as above, an irreducible, non–degenerate, projective variety of dimenszion $n$. For every non–negative integer $k$, we set:

$$r_k = r_k(X) := r - \dim(T_{X,p_1,\dots,p_k}) - 1 = r - s^{(k-1)}(X) - 1.$$

Consider the projection of $X$ with centre $T_{X,p_1,\dots,p_k}$. We call this a *general $k$–tangential projection* of $X$, and we will denote it by $\tau_{X,p_1,\dots,p_k}$ or simply by $\tau_{X,k}$. We will denote by $X_k$ its image. Notice that $X_k$ is non–degenerate in $\mathbb{P}^{r_k}$. We define $\tau_{X,0}$ as the identity so that $X_0 = X$.

We set $n_k := n_k(X) := \dim(X_k)$ and $m_k := m_k(X) = n - n_k$. Notice that $m_k$ is the dimension of the general fibre of the map $\tau_{X,k} : X \dashrightarrow X_k$.

LEMMA 1.14. *Let $X \subset \mathbb{P}^r$ be an irreducible, projective variety of dimension $n \geq 2$. Then $m_k \leq \gamma_k$.*

*Proof.* Consider the general $k$–tangential projection $\tau_{X,k} : X \dashrightarrow X_k$ from $T_{X,p_1,\dots,p_h}$. Let $p_0$ be a general point of $X$. The pull–back via $\tau_{X,k}$ of the tangent space to $X_k$ at $\tau_{X,h}(p_0)$ is $T_{X,p_0,\dots,p_h}$. Hence $T_{X,p_0,\dots,p_k}$ is tangent to $X$ along the whole fibre $\tau_{X,k}^{-1}(\tau_{X,k}(p_0))$, which has dimension $m_k$. By the definition of $\gamma_k$ we have the assertion. ∎

The following result is an immediate consequence of Terracini's lemma (see [6], §3):

PROPOSITION 1.15. *Let $X \subset \mathbb{P}^r$ be an irreducible, projective variety of dimension $n \geq 2$ which is minimally $k$–defective. Then:*
  (i) *$n_h = n$ for $1 \leq h \leq k - 1$, whereas $n_k \leq n - \delta_k < n$, thus $m_h = 0$ for $1 \leq h \leq k - 1$, whereas $m_k \geq \delta_k$;*
  (ii) *$0 < n_k < r_k$, i.e. $X_k$ is a proper subvariety of positive dimension of $\mathbb{P}^{r_k}$;*
  (iii) *if $r \geq n(k+1) + k$ then $n_k = n - \delta_k$, i.e. $m_k = \delta_k$;*
  (iv) *$\delta_k \leq n - 1$ and $r \geq (n+1)k + 2$;*
  (v) *if $r = (n+1)k + 2$ then $\delta_k = 1$, $m_k = n - 1$ and $X_k$ is a plane curve.*

*Proof.* Let us prove part (i). Consider the general $h$–tangential projection $\tau_{X,h} : X \dashrightarrow X_h \subseteq \mathbb{P}^{r_h}$ from $T_{X,p_1,\dots,p_h}$. Let $p_0$ be a general point of $X$. For all $h$, the pull–back via $\tau_{X,h}$ of the tangent space to $X_h$ at $\tau_{X,h}(p_0)$ is $T_{X,p_0,\dots,p_h}$, hence:

$$(6) \qquad s^{(h)} = \dim(T_{X,p_0,\dots,p_h}) = n_h + \dim(T_{X,p_1,\dots,p_h}) + 1 = n_h + s^{(h-1)} + 1.$$

Since $X$ is minimally $k$–defective, then for all $i < k$ one has $s^{(i)} = (i+1)n + i$. Hence formula (6) gives, for all $h < k$, $n_h = n$.

On the other hand, we know that:

$$(7) \qquad\qquad \dim(T_{X,p_0,\dots,p_k}) = s^{(k)} < \sigma^{(k)} \leq n(k+1) + k.$$

Hence by formula (6), applied for $h = k$, we have that $n_k \leq n - \delta_k < n$. This proves (i).

Since $X$ is $k$-defective, then $\dim(S^k(X)) < r$, hence $\dim(T_{X,p_0,\dots,p_k}) = s^{(k)} < r$, i.e. $T_{X,p_0,\dots,p_k}$ cannot coincide with the whole space. Therefore $X_k$ is a proper subvariety of $\mathbb{P}^{r_k}$. Since it is non–degenerate, one has $n_k > 0$. This proves part (ii).

If $r \geq n(k+1)+k$ then $\sigma^{(k)} = n(k+1)+k$. Therefore (6) and (7) yield $n_k = n - \delta_k$, i.e. part (iii).

Since $n_k > 0$ one has $\delta_k < n$. Furthermore $r - (n+1)k = r_k > n_k \geq 1$. Hence $r \geq (n+1)k + 2$. This proves part (iv).

If $r = (n+1)k+2$, then $\sigma^{(k)} = \min\{n(k+1)+k, r\} = r = (n+1)k+2$ and formula (6), applied for $h = k$, gives $n_k = 2 - \delta_k$. On the other hand $2 = r - (n+1)k = r_k > n_k \geq 1$, thus $n_k = 1$ and $\delta_k = 1$. This proves part (v). ∎

The following example is well known:

EXAMPLE 1.16. Let $k \geq 1$ be an integer and let $X$ be the Segre embedding of $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$ in $\mathbb{P}^{k^2+4k+3}$. We claim that $X$ is 1–defective with $\delta_1(X) = 2$, $n_1(X) = 2k$.

Indeed one sees that $X_1$ is nothing but the Segre embedding of $\mathbb{P}^k \times \mathbb{P}^k$ in $\mathbb{P}^{k^2+2k}$. The assertion immediately follows.

1.11.   We recall the following elementary criterion, needed in the sequel, which tells us when a variety sits in a cone (see [6], Proposition 4.1):

PROPOSITION 1.17. *Let $X \subset \mathbb{P}^r$ be an irreducible, projective variety of dimension $n$. Let $\Pi \subset \mathbb{P}^r$ be a projective subspace of dimension $s$ not containing $X$. Then $X$ projects from $\Pi$ to a variety $Y$ of dimension $m < n$, i.e. $X$ sits in the cone with vertex $\Pi$ over $Y$, if and only if the general tangent space to $X$ intersects $\Pi$ along a subspace of dimension $n - m - 1$. In particular $X$ is a cone with vertex $\Pi$ if and only if the general tangent space to $X$ contains $\Pi$ and $X$ sits in a $\mathbb{P}^{s+1}$ containing $\Pi$ if and only if the general tangent space to $X$ meets $\Pi$ along a a subspace of dimension $n - 1$.*

As a consequence we have:

PROPOSITION 1.18. *Let $X \subset \mathbb{P}^r$ be an irreducible, projective variety of dimension $n$. Let $\Pi \subset \mathbb{P}^r$ be a projective subspace of dimension $s \geq 0$ not containing $X$, let $\Pi'$ be a complementary subspace of dimension $r - s - 1$, and assume that $X$ projects from $\Pi$ to a variety $Y \subset \Pi'$ of dimension $n - 1$. Then:*

   (i) *for every $k \geq s$, $S^k(X)$ is the cone with vertex $\Pi$ over $S^k(Y)$, and therefore $s^{(k)}(X) = s^{(k)}(Y) + s + 1$ and $\gamma_k(X) = \gamma_k(Y) + 1$. In particular $X$ is $k$–defective if $k > s$ [and $\gamma_k(X) = 1$ if and only if $Y$ is not $k$–weakly defective] and $X$ is $s$–defective if and only if $Y$ is $s$–defective;*

   (ii) *for every positive integer $k < s$, one has $s^{(k)}(X) \geq s^{(k)}(Y) + k + 1$. In particular if $Y$ is not $k$–defective, then the equality holds and $X$ is also not $k$–defective.*

*As a consequence, if $Y$ is not $s$–defective, then $X$ is minimally $(s+1)$–defective, whereas if $Y$ is minimally $s$–defective, then also $X$ is minimally $s$–defective. Finally then $\gamma_k(X) = 1$ if and only if $Y$ is not $(s+1)$–weakly defective.*

*Proof.* Let $p \in X$ be a general point. By Proposition 1.17, $T_{X,p}$ meets $\Pi$ in one point. Hence there is a map:

$$f : p \in X \dashrightarrow T_{X,p} \cap \Pi \in \Pi$$

and the image of $f$ is non–degenerate in $\Pi$ since $X$ is non–degenerate in $\mathbb{P}^r$.

Let $k \geq s$. Let $p_0, ..., p_k$ be general points of $X$ and let $q_0, ..., q_k$ be the corresponding projections on $Y$. Then $T_{X,p_0,...,p_k}$ contains $\Pi$ and projects from $\Pi$ to $T_{Y,q_0,...,q_k}$. Hence the general tangential $k$–contact locus of $S$ pulls–back to $X$ to the general tangential $k$–contact locus of $X$. Moreover, by Terracini's lemma and by Proposition 1.17, we have that $S^{(k)}(X)$ is a cone with vertex $\Pi$. By Proposition 1.12, $S^{(k)}(X)$ is the cone with vertex $\Pi$ over $S^{(k)}(Y)$. Thus (i) follows.

Let $k \leq s$ be a positive integer. The above argument shows that $T_{X,p_0,...,p_k}$ intersects $\Pi$ along a $\mathbb{P}^l$, with $l \geq k$. Since $S^k(X)$ projects from $\Pi$ to $S^k(Y)$, part (ii) follows from Proposition 1.17.

The rest of the assertion is trivial. ∎

In a similar way one proves the following:

PROPOSITION 1.19. *Let $X \subset \mathbb{P}^r$ be an irreducible, projective variety of dimension $n$. Let $\Pi \subset \mathbb{P}^r$ be a projective subspace of dimension $2s \geq 0$ not containing $X$ and assume that $X$ projects from $\Pi$ to a variety $Y \subset \mathbb{P}^{r-2s-1}$ of dimension $n - 2$. Then:*

(i) *for every $k \geq s$, $S^{(k)}(X)$ is the cone with vertex $\Pi$ over $S^{(k)}(Y)$, hence $s^{(k)}(X) = s^{(k)}(Y) + 2s + 1$ and therefore $X$ is $k$ defective;*

(ii) *for every positive integer $k < s$, one has $s^{(k)}(X) \geq s^{(k)}(Y) + 2k + 2$. In particular if $Y$ is not $k$–defective, then the equality holds and $X$ is also not $k$–defective.*

*As a consequence, if $Y$ is not $(s-1)$–defective, then $X$ is minimally $s$–defective.*

PROPOSITION 1.20. *Let $X \subset \mathbb{P}^r$ be an irreducible, projective variety of dimension $n$. Let $\Pi \subset \mathbb{P}^r$ be a projective subspace of dimension $2s - 1 \geq 0$ not containing $X$ and assume that $X$ projects from $\Pi$ to a variety $Y \subset \mathbb{P}^{r-2s}$ of dimension $n - 2$. Then:*

(i) *for every $k \geq s - 1$, $S^{(k)}(X)$ is the cone with vertex $\Pi$ over $S^{(k)}(Y)$, hence $s^{(k)}(X) = s^{(k)}(Y) + 2s$. In particular $X$ is $k$ defective if $k > s - 1$ and $X$ is $(s-1)$–defective if and only if $Y$ is $(s-1)$–defective;*

(ii) *for every positive integer $k < s - 1$, one has $s^{(k)}(X) \geq s^{(k)}(Y) + 2k + 2$. In particular if $Y$ is not $k$–defective, then the equality holds and $X$ is also not $k$–defective.*

*As a consequence, if $Y$ is not $(s-1)$–defective, then $X$ is minimally $s$–defective.*

1.12. Now we give a definition which extends a definition given in [22].

DEFINITION 1.21. Let $Z \subseteq \mathbb{P}^r$ be an irreducible, non–degenerate, projective variety of dimension $n$. Let $h$ be a non–negative integer such that $h \leq n$. Let $z \in Z$ be a general point and suppose that $\overline{T_{Z,z} \cap Z - \{x\}}$ has an irreducible component of dimension $h$. In this case we will say that $Z$ is $h$–*tangentially degenerate*. We will simply say that $Z$ is *tangentially degenerate* if it is $h$–tangentially degenerate for some $h$.

Remark that, if $h > 0$, to say that $Z$ is $h$–tangentially degenerate is equivalent to say that $T_{Z,z} \cap Z$ has an irreducible component of dimension $h$.

We notice the following results:

PROPOSITION 1.22. *Let $Z \subseteq \mathbb{P}^r$ be an irreducible, non–degenerate, projective variety of dimension $n$. One has:*
*(a) $Z$ is $n$–tangentially degenerate if and only if $r = n$ and $Z = \mathbb{P}^n$;*
*(b) if $n \geq 2$, then $Z$ is $(n-1)$–tangentially degenerate if and only if either $r = n+1$ or $Z$ is a scroll over a curve.*

*Proof.* Part (a) is obvious. Part (b) is classical, and it can be easily deduced from [13], Theorem 2.1 and Theorem 3.2. ∎

LEMMA 1.23. *Let $X \subset \mathbb{P}^r$, be an irreducible, projective variety, with $r > 2n + 1$. Let $x \in X$ be a general point and suppose that the projection of $X$ from $x$ is a variety $X' \subset \mathbb{P}^{r-1}$ which is not tangentially degenerate. Then a general tangential projection of $X$ is a birational map to its image.*

*Proof.* If the conclusion does not hold, then for $x, y \in X$ general points, the $(n+1)$–dimensional space spanned by $x$ and $T_{X,y}$ meets $X$ at another point $x'$. But then, by projecting $X$ from $x$, we get a variety $X' \subset \mathbb{P}^{r-1}$ such that the tangent space to $X'$ at a general point $z$ meets $X'$ in some other point $z'$. ∎

1.13.    By Terracini's lemma and elementary considerations (see Proposition 1.22), there are no defective curves. The study of defective surfaces goes back to Palatini [24], Scorza [27], Terracini [30], whose classification result is the first complete one. In modern times, we mention Dale [11] and Catalano–Johnson [4], whereas in [6] one finds the full classification of $k$–weakly defective surfaces for any $k$, which includes Terracini's classification of defective surfaces.

The classification of 1–defective threefolds was taken up by Scorza in [26]. The case of smooth 1–defective threefolds was also examined by Fujita [16] and Fujita–Roberts [17]. Scorza's classification has been revisited in [7]. We recall here the result:

THEOREM 1.24. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, 1–defective threefold. Then $r \geq 6$ and $X$ is of one of the following types:*

*(1) $X$ is a cone over a surface $S$;*
*(2) $X$ sits in a 4–dimensional cone over a curve $C$;*
*(3) $r = 7$ and $X$ sits in a 4–dimensional cone over the Veronese surface in $\mathbb{P}^5$;*
*(4) $X$ is either the 2–Veronese embedding of $\mathbb{P}^3$ in $\mathbb{P}^9$ or a projection of it in $\mathbb{P}^r$, $r = 7, 8$;*
*(5) $r = 7$ and $X$ is a hyperplane section of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$.*

*Conversely, all the irreducible threefolds in the above list are 1-defective.*

*Furthermore $\delta_1 \leq 2$ and $m_1 \leq 2$ and actually $m_1 = \delta_1 = 1$ unless we are in case (1) and $X$ is either a cone over a curve or a cone over a Veronese surface in $\mathbb{P}^5$, in which case $m_1 = \delta_1 = 2$.*

REMARK 1.25. It is useful to remark that, for a 1–defective threefold $X$ as in the various cases listed in Theorem 1.24 above, the general tangential 1–contact locus is:

(1) a reducible curve consisting of two general rulings of $X$, unless $S$ is a 1–weakly defective surface;
(2) a reducible surface consisting of two general fibres of the projection of $X$ to $C$;
(3) an irreducible surface, the pull–back on $X$ of a general conic of the Veronese surface in $\mathbb{P}^5$;
(4) an irreducible conic, the image of a general line in $\mathbb{P}^3$;
(5) an irreducible conic ([7], Example 2.5).

Notice that:

- if the general tangential 1–contact locus is a reducible curve, then we are in case (1) and $X$, being a cone over a non–defective surface, lies in $\mathbb{P}^r$, with $r \geq 7$;
- if the general tangential 1–contact locus is a reducible surface, then we are either in case (2) or in case (1) and $X$ is the cone over a 1–weakly defective surface with reducible general tangential 1–contact locus. By looking at the classification of weakly defective surfaces (see [6], Theorem 1.3), we see that also in this latter case $X$ sits in a cone of dimension (4) over a curve and that $X$ sits in $\mathbb{P}^r$, with $r \geq 7$;
- if the general tangential 1–contact locus is an irreducible curve, then we are in case (4) or (5). Let $\mathcal{G}$ the family of tangential 1–contact loci. This is a 4–dimensional family of conics. If $x \in X$ is a general point, we have therefore a 2–dimensional family $\mathcal{G}_x$ of conics in $\mathcal{G}$ passing through $x$. It is important to notice, for future purposes, that the tangent lines to the conics in $\mathcal{G}_x$ fill up the whole tangent space $T_{X,x}$. This is trivial in case (4), and in case (5) it immediately follows by the discussion in [7], Example 2.5.

In connection with the privious classification results, it is worth pointing out the following:

PROPOSITION 1.26. Let $X \subset \mathbb{P}^r$, $r \geq (k+1)n + k$, be an irreducible, non–degenerate projective variety. Suppose that $X$ enjoyes the property that $k+1$ general points of $X$ lie on a curve $\Gamma \subset X$ and such that $\dim(< C >) \leq 2k$. Then $X$ is $k$–defective.

*Proof.* Notice that $\Gamma$ is not defective and therefore $S^k(\Gamma) = < \Gamma >$. Furthemore there are infinitely many $(k+1)$–secant $\mathbb{P}^k$'s to $\Gamma$ and therefore to $X$, containing the general point of $< \Gamma >$. This shows that there are infinitely many $(k+1)$–secant $\mathbb{P}^k$'s to $X$ containing the general point of $S^k(X)$, thus $X$ is $k$–defective. ∎

## 2. BASIC PROPERTIES OF DEFECTIVE THREEFOLDS

From now on we will concentrate on defective threefolds. In this section we make a remark which, though not really needed in the sequel, is, in our opinion, of some interest.

We start by noticing that the basic tool for the proof of Theorem1.24 is the study of the first general tangential projection $\tau_{X,1} : X \dashrightarrow X_1$. One studies separately the cases in which $X_1$ is either a curve or a weakly–defective surface. In these situations, one has interesting information about the general hyperplane section of $X$ which are important steps toward the classification. When $k > 1$, the corresponding analysis is not conclusive. However, one has the following results:

PROPOSITION 2.1. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, minimally $k$–defective threefold. Assume that $m_k = 2$, i.e. $X_k$ is a curve. Let $S$ be a general hyperplane section of $X$. Then the general $k$–tangential projection $S_k$ of $S$ sits in a cone with a vertex of dimension at most $k - 2$ over the curve $X_k$.*

*Moreover:*

    (i) *when $r \geq 6k$, then $S$ is $(2k - 1)$–defective;*
    (ii) *when $r \geq 6k - 2$, then $S$ is $(2k - 1)$–weakly defective.*

*Proof.* Since $X$ is $k$–defective, we have that $S^k(X)$ is a proper subvariety of $\mathbb{P}^r$, hence $k < r$. Let $p_0, ..., p_k$ be general points on $X$. Then $< p_0, ..., p_k >$ is a proper subspace of $\mathbb{P}^r$. Let $H$ be a general hyperplane containing $< p_0, ..., p_k >$. By varying the points $p_0, ..., p_k$ on $X$, then $H$ also varies and it turns out to be a general hyperplane in $\mathbb{P}^r$. Let $S$ be the surface cut out by $H$ on $X$. Then:

$$(8) \qquad \dim(H \cap T_{X,P_1,...,P_k}) - \dim(T_{S,P_1,...,P_k}) \leq k - 1.$$

Notice that $S_k$, which is the projection of $S$ from $T_{S,P_1,...,P_k}$, is contained in a cone $W$ over the projection $S'$ of $S$ from $H \cap T_{X,P_1,...,P_k}$. We may assume that $W$ is of minimal dimension with the property of containing $S_k$. The vertex $\Pi$ of $W$ is contained in the projection of $H \cap T_{X,P_1,...,P_k}$ from $T_{S,P_1,...,P_k}$, hence, by (8), we have $\dim(\Pi) \leq k - 2$. Remark that $S'$ is also the projection of $S$ from $T_{X,p_1,...,p_k}$. Hence $S'$ is contained in $X_k$. Since $X_k$ is a curve, we have $S' = X_k$, proving the first part of the assertion.

The tangent space to $S_k$ at a point $q$ intersects in a point the vertex $\Pi$ of the cone $W$, because the projection of $S_k$ from $\Pi$ is a curve (see Proposition 1.17). Let $q_1, \ldots, q_{k-1}$ be general points of $S_k$. Then, by the minimality assumption on $W$, the space $T_{S_k,Q_1,...,Q_{k-1}}$ contains $\Pi$. Thus the general $(k - 1)$–tangential projection of $S_k$, which is the general $(2k - 1)$–tangential projection of $S$, certainly has dimension at most 1.

Assume that $S$ is not $k$–defective, otherwise there is nothing to prove. Then $S_k$ sits in $\mathbb{P}^s$, $s = r - 3k - 1$. From the previous argument, we get that the span of $k$ general tangent planes to $S_k$ has dimension at most $3k - 2$. Thus, if $s \geq 3k - 1$, i.e. if $r \geq 6k$, then $S_k$ is $(k - 1)$–defective, hence $S$ is $(2k - 1)$–defective. This proves part (i).

If the span of $k - 1$ general tangent planes to $S_k$ has dimension at most $3k - 5$ and if $s \geq 3k - 4$, i.e. if $r \geq 6k - 3$, then $S_k$ is $(k - 2)$–defective, hence $S$ is $(2k - 2)$–defective and therefore also $(2k - 1)$–defective.

Assume that the span of $k - 1$ general tangent planes to $S_k$ has dimension $3k - 4$. Then the projection of $S_k$ from $T_{S_k, q_1, \ldots, q_{k-2}}$ is a surface and therefore $T_{S_k, q_1, \ldots, q_{k-2}}$ cannot contain the vertex $\Pi$ of the cone $W$. Since, as we saw, all spaces $T_{S_k, q_i}$ intersect $\Pi$ in a point, by the minimality assumption on $W$ we have that $T_{S_k, q_1, \ldots, q_{k-2}} \cap \Pi$ has codimension 1 in $\Pi$. Now the projection $S_{2k-2}$ of $S_k$ from $T_{S_k, q_1, \ldots, q_{k-2}}$ is a surface which sits in the cone $Z$ over $X_k$ with vertex the projection of $\Pi$ from $T_{S_k, Q_1, \ldots, Q_{k-2}} \cap \Pi$, which is a point. Hence $S_{2k-2}$ coincides with the cone $Z$. Moreover $S_{2k-2}$ spans a $\mathbb{P}^{s'}$ where $s' = s - 3(k-2) = r - 3k - 1 - 3(k-2) = r - 6k + 5$. If $s' \geq 3$, i.e. if $r \geq 6k - 2$, then $S_{2k-2}$ is 1-weakly defective. Then $S$ is $(2k-1)$-weakly defective (see [6], Proposition (3.6)). ∎

Similar arguments lead to the following:

PROPOSITION 2.2. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, minimally $k$–defective threefold. Assume that $m_k = 1$ and that $X_k$ is a developable surface. Let $S$ be a general hyperplane section of $X$. Then the general $k$-tangential projection $S_k$ of $S$ sits in a cone of dimension $k+1$ over the developable surface $X_k$. When $r \geq 6k + 1$, then $S$ is $(2k - 1)$-weakly defective.*

## 3. BASIC PROPERTIES OF THE CONTACT LOCI

In this section we study, mostly in the case of defective threefoolds, some basic properties of the contact loci $\Gamma$ and $\Sigma$ defined in §1.

We start with the following:

PROPOSITION 3.1. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, $k$–defective variety. For a general choice of $p_0, \ldots, p_k \in X$ and a general choice of the hyperplane $H$ containing $T_{X, p_0, \ldots, p_k}$, the contact loci $\Gamma = \Gamma_{p_0, \ldots, p_k}$ and $\Sigma = \Sigma(H) = \Sigma_{p_0, \ldots, p_k}(H)$ are equidimensional and smooth at each of the points $p_0, \ldots, p_k$. Furthermore either they are irreducible or they consist of $k + 1$ irreducible components, one through each of the points $p_0, \ldots, p_k$.*

*Proof.* We prove the assertion for $\Gamma$. The proof for $\Sigma$ is quite similar.

First of all, let us move slightly the points $p_i$'s on some component of $\Gamma$ to a new set of points $\{q_0, \ldots, q_k\}$. Then $q_0, \ldots, q_k$ are also general points on $X$. Furthermore $T_{X, p_0, \ldots, p_k}$ contains the tangent spaces to $X$ at the points $q_i$'s, so for dimension reasons, it coincides with $T_{X, q_0, \ldots, q_k}$. Hence $T_{X, q_0, \ldots, q_k}$ is also tangent to $X$ along $\Gamma = \Gamma_{p_0, \ldots, p_k}$. This tells us that, by the generality of the points $p_0, \ldots, p_k$, $\Gamma$ is smooth at $p_0, \ldots, p_k$, and therefore there is only one irreducible component of $\Gamma$ through each of the points $p_0, \ldots, p_k$. Since $[p_0, \ldots, p_k]$ is a general point of $\mathrm{Sym}^{k+1}(X)$, there is the monodromy action of the full symmetric group $\mathcal{S}_{k+1}$ on $p_0, \ldots, p_k$ as recalled in §1.5. Hence we can permute the points $p_i$ as we like, and this implies that all components of $\Gamma$ have the same dimension.

Assume now that there is a component of $\Gamma$ which contains more than one of the points $p_i$'s, say $p_0$ and $p_1$. Again by monodromy, we can let $p_0$ stay fixed and we can move $p_1$ to any one of the points $p_i$, $i > 1$. Then we see that also $p_0$ and $p_i$, $i > 1$, stay on an irreducible component of $\Gamma$. Since $p_0$ sits on only one irreducible component of $\Gamma$, then this component has to contain all the points $p_0, \ldots, p_k$ and therefore it has to coincide with $\Gamma$. This proves the proposition. ∎

Since $\Gamma \subseteq \Sigma$, the next corollary immediately follows:

COROLLARY 3.2. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, $k$– defective variety of dimension $n$. If $\epsilon_k = \gamma_k$, e.g. if either $\epsilon_k = 1$ or $\gamma_k = n - 1$, then the general contact locus $\Sigma$ coincides with the general tangential contact locus $\Gamma$.*

The next proposition, which will be useful later, tells us how tangential contact loci behave under a general tangential projection:

PROPOSITION 3.3. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, $k$– defective, but not $1$–defective variety of dimension $n$, hence $k \geq 2$. Let $X_1$ be the general tangential projection of $X$. Then the general tangential $k$–contact locus $\Gamma$ of $X$ and the general tangential $(k-1)$–contact locus $\Gamma_1$ of $X_1$ have the same dimension and $\Gamma$ is reducible if and only if $\Gamma_1$ is.*

*Proof.* Since $X$ is not $1$–defective, we have $n_1 = n$. Hence, if $p_0 \in X$ is a general point, the projection $\tau := \tau_{X,1} : X \dashrightarrow X_1$ from $T_{X,p_0}$ is generically finite. If $p_1, \ldots, p_k$ are general points of $X$, so that $\tau(p_1), \ldots, \tau(p_k)$ are general points of $X_1$, then the image of the span $T_{X,p_0,p_1,\ldots,p_k}$ via $\tau$ is the span $T_{X_1,\tau(p_1),\ldots,\tau(p_k)}$. Hence $\Gamma$ maps onto $\Gamma_1$ via $\tau$. The generic finiteness of $\tau$ implies $\dim(\Gamma) = \dim(\Gamma_1)$. Moreover if $\Gamma_1$ is reducible then also $\Gamma$ is.

Conversely, suppose $\Gamma = \Gamma_{p_0,\ldots,p_k}$ is reducible. Then $\Gamma = \cup_{i=0}^{k}\Gamma^{(i)}$, where $\Gamma^{(i)}$ is the irreducile component containing the point $i_i$. Set $\Gamma_1^{(i)} = \tau(\Gamma^{(i)})$, so that $\Gamma_1^{(i)}$ is the component of $\Gamma_1$ through the point $\tau(p_i)$. Since $p_1, \ldots, p_k$ are general points of $X$, then $\tau(p_1), \ldots, \tau(p_k)$ are general points of $X_1$. Therefore, once $p_1$, and thus $\Gamma_1^{(1)}$, has been fixed, we can choose $p_2$ so that $\tau(p_2) \notin \Gamma_1^{(1)}$, and so $\Gamma_1^{(1)} \neq \Gamma_1^{(2)}$. Proceeding in this way we see that for a general choice of $p_1, \ldots, p_k$, the varieties $\Gamma_1^{(1)}, \ldots, \Gamma_1^{(k)}$ are all distinct varieties, proving that $\Gamma_1$ is reducible. ∎

Now we restrict to the threefold case. One can easily detect when $\Gamma$, and therefore $\Sigma$, is a divisor, from the behaviour of the tangential projection of $X$:

PROPOSITION 3.4. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, min- imally $k$–defective threefold. Then $\gamma_k = 2$ if and only if either one of the following holds:*

(i) *$X_k$ is a curve;*
(ii) *$X_k$ is a developable surface.*

*Proof.* We know that $0 < n_k < 3$. Suppose that $\gamma_k = 2$. Choose general points $p_0, \ldots, p_k \in X$. Let $\Gamma_0$ be the irreducible component of $\Gamma_{p_0,\ldots,p_k}$ containing $p_0$. Consider the projection $\tau_{X,k}$ from the space $T_{X,p_1,\ldots,p_k}$. Then $\tau_{X,k}(\Gamma_0)$ is contained in $\tau_{X,k}(T_{X,p_0,\ldots,p_k})$, which is a projective space of dimension $n_k$. This shows that $\tau_{X,k}(\Gamma_0)$ is a proper subvariety of $X_k$, otherwise $X_k$ would be equal to the linear space $\tau_{X,k}(T_{X,p_0,\ldots,p_k})$, whereas we know that $X_k$ is a proper, non–degenerate subvariety of $\mathbb{P}^{r_k}$ (see Proposition 1.15). Hence $\tau_{X,k}(\Gamma_0)$ is either a point or a curve. In the former case the fiber of $\tau_{X,k}$ at $p_0$, which is a general point of $X$, contains a divisor, hence $n_k = 1$. In the latter case, the tangent plane to $X_k$ at its general point $\tau_{X,k}(p_0)$ meets $X_k$ along the curve $\tau_{X,k}(\Gamma_0)$, which passes through $\tau_{X,k}(p_0)$. Hence $X_k$ is a developable surface.

Conversely if either $n_k = 1$ or $X_k$ is a developable surface, then $\dim(\Gamma) = 2$, because $\Gamma$ contains the pull–back, via $\tau_{X,k}$, of the contact locus of $X_k$ with its general tangent space. ∎

Next, we look at the families $\mathcal{G}$ and $\mathcal{S}$ respectively describing the general contact loci $\Gamma = \Gamma_{p_0,\ldots,p_k}$ when $p_0, \ldots, p_k$ vary and by $\Sigma = \Sigma(H)_{p_0,\ldots,p_k}$ when $p_0, \ldots, p_k$ and $H$ vary. Recalling the definition of involution from §1.6, we have:

LEMMA 3.5. *In the above setting, if $\gamma_k = 2$ [resp. $\epsilon_k = 2$], then the family of divisors $\mathcal{G}$ [resp. $\mathcal{S}$] is an involution of dimension $k + 1$. Hence, if the general member of $\mathcal{G}$ [resp. of $\mathcal{S}$] is irreducible, then $\mathcal{G}$ [resp. $\mathcal{S}$] is a linear system.*

*Proof.* To prove the first part of the assertion one argues as in [6], pp. 172–173: though one refers to the surface case there, the argument applies, word by word, to our situation. The final part of the assertion follows by Theorem 1.9. ∎

The following proposition gives more precise information about the situation described in Proposition 3.4:

PROPOSITION 3.6. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, minimally $k$–defective threefold. Assume that $\gamma_k = 2$. Then:*

(i) *when $X_k$ is a curve and $r > 4k + 2$, then the involution $\mathcal{G}$ is composed with a pencil;*

(ii) *when $X_k$ is a developable surface, the same conclusion holds provided $r > 4k + 3$.*

*Proof.* Assume that $r > 4k + 2$ and $X_k$ is a curve. By Proposition 1.15, we know that $X_{k-1}$ is a threefold in $\mathbb{P}^{r_{k-1}}$, whose general tangential projection is a curve. Notice that $r_{k-1} = r + 4 - 4k > 6$. Then by Theorem 1.24 we have that $X_{k-1}$ is a cone over a curve with vertex a line and therefore two general tangent spaces to $X_{k-1}$ have contact with $X_{k-1}$ along two planes. By pulling back to $X$, we see that $\Gamma_{p_0,\ldots,p_k}$ has at least two irreducible components. It follows from Theorem 1.9 that $\mathcal{G}$ is composed with a pencil. This finishes case (i).

Similarly, assume that $X_k$ is a develobable surface and $r > 4k + 3$. By Lemma 3.6 of [7] a general hyperplane section $S'$ of $X_{k-1}$ is 1–weakly defective but not 1–defective. By the classification of weakly defective surfaces (see [6], Theorem (1.3)), one gets that either:

(a) $S'$ is developable or
(b) $S'$ is contained in a cone over a curve, with vertex along a line or
(c) $S'$ sits in $\mathbb{P}^6$.

Case (c) is excluded because $S'$ is non–degenerate in $\mathbb{P}^{r_{k-1}-1}$ and $r_{k-1} - 1 = r + 3 - 4k > 6$. Then only cases (a) and (b) may occur and, in both of them the tangential contact variety $\Gamma'$ of $S'$ with the span of two general tangent planes is reducible.

Now we claim that the general tangential contact locus of $X_{k-1}$ at two general points is reducible. This, as in case (i), leads to the assertion. As for the claim, let $q_1, q_2$ be two general points on $X_{k-1}$. Then $T_{X_{k-1},q_1,q_2}$ is a $\mathbb{P}^6$, because $\delta_k(X) = \delta_1(X_{k-1}) = 1$. Taking a general hyperplane $H'$ through $q_1, q_2$ and its section $S'$ with $X_{k-1}$, we have that $T_{S',q_1,q_2}$ is a $\mathbb{P}^5$, because $S'$ is not 1–defective. Hence $T_{S',q_1,q_2} = H' \cap T_{X_{k-1},q_1,q_2}$. Let $\overline{\Gamma}$ be the contact variety of $T_{X_{k-1},q_1,q_2}$ with $X_{k-1}$. Of course the tangential contact locus $\Gamma'$ for $S'$ is the intersection of $\overline{\Gamma}$ with $H'$. Since $\Gamma'$ is reducible and by the genericity of $H'$, we see that $\overline{\Gamma}$ is reducible. This proves the claim and therefore the proposition. ∎

## 4. THE IRREDUCIBLE DIVISORIAL CASE

Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, minimally $k$–defective threefold with $k \geq 2$. In this section we examine the case in which $\gamma_k(X) = 2$ and the general tangential $k$–contact locus is irreducible.

By Proposition 3.4 we know that either $X_k$ is a curve or $X_k$ is a developable surface. Furthermore Proposition 3.6 and Proposition 1.15 imply that $4k + 2 \leq r \leq 4k + 3$ and if $r = 4k + 2$ then $X_k$ is a curve.

Recall that, by Lemma 3.5, the family of divisors $\mathcal{G}$ of tangential $k$–contact loci is a linear system of dimension $k + 1$ which is not composed with a pencil.

Observe that, by construction, the linear system $2\mathcal{G}$ (see §1.4) is contained in the hyperplane linear system $\mathcal{H}$ of $X$. In particular we have the linear equivalence relation $H \equiv 2\Gamma + \Delta$, with $\Delta$ effective.

We let:

$$\phi_{\mathcal{G}} : X \dashrightarrow Y \subset \mathbb{P}^{k+1}$$

be the rational map defined by $\mathcal{G}$. Since $\mathcal{G}$ is not composed with a pencil, then $n := \dim(Y) > 1$. We set $d := \deg(Y)$. The classification is based on the classification of $Y \subset \mathbb{P}^{k+1}$ according to its dimension and degree. A straightforward application of Theorem 1.1 gives the following useful information:

LEMMA 4.1. *In the above setting, one has:*

$$(9) \qquad 4k + 4 \geq r + 1 \geq h_Y(2) \geq \iota + k(n+1) - \frac{d(d-3)}{2} + 2$$

*where $\iota = \iota(Y) = \min\{d - k + n - 2, k + 1 - n\}$. Moreover if $d < 2(k - n) + 3$, the equality holds in (9) if and only if the general curve section of $Y$ is projectively normal.*

We will discuss separately the two cases $n = 2, 3$.

THEOREM 4.2. *In the above setting, assume that $n = 3$. Then $\iota \leq 3$, $\Delta = 0$ and we are in one of the following cases:*

(1) *$\iota = 0$ and $Y$ is a threefold of minimal degree in $\mathbb{P}^{k+1}$; then $X$ is contained in a cone over the 2–uple embedding of $Y$, with vertex either a point if $r = 4k+2$ or a line if $r = 4k + 3$;*

(2) *$\iota = 1$, $k = 3$, and $Y$ is a hypersurface of degree $d \geq 3$ in $\mathbb{P}^4$; then either $r = 4k + 2 = 14$ and $X$ is the 2-uple embedding of $Y$ or $r = 4k + 3 = 15$ and $X$ is contained in the cone with vertex a point over the 2-uple embedding of $Y$;*

(3) *$\iota = 1$ and $Y$ is a threefold of degree $k$ in $\mathbb{P}^{k+1}$ with curve sections of arithmetic genus 1; then either $r = 4k + 2$ and $X$ is the 2-uple embedding of $Y$ or $r = 4k+3$ and $X$ is contained in the cone with vertex a point over the 2-uple embedding of $Y$;*

(4) *$\iota = 1$ and $Y$ is a threefold of degree $k$ in $\mathbb{P}^{k+1}$ with curve sections of genus 0 which is either a cone with vertex a line over a smooth rational curve of degree $k$ in $\mathbb{P}^{k-1}$ or it has a double line; then $r = 4k+3$ and $X$ is the 2-uple embedding of $Y$;*

(5) *$\iota = 2$, $k = 4$ and $Y$ is a threefold of degree $d \geq 5$ in $\mathbb{P}^5$, contained in a quadric; then $r = 4k + 3 = 19$ and $X$ is the 2-uple embedding of $Y$;*

(6) *$\iota = 2$, $k \geq 4$ and $Y$ is a threefold of degree $k + 1$ in $\mathbb{P}^{k+1}$ with curve sections of arithmetic genus 2; then $r = 4k + 3$ and $X$ is the 2-uple embedding of $Y$.*

*Proof.* In the present case equation (9) reads:

$$4k + 4 \geq r + 1 \geq h_Y(2) \geq \iota + 4k + 2$$

hence $0 \leq \iota \leq 2$. Then $2\mathcal{G}$ is a subsystem of $\mathcal{H}$ of codimension $\leq 2$. It follows that $\Delta$ imposes at most two conditions to the hyperplanes of $\mathbb{P}^{k+1}$. Since $\Delta$ is a divisor, the only possibility is that $\Delta = 0$.

If $\iota = 0$, then Theorem 1.1 yields $d = k-1$, i.e. $Y$ is a threefold of minimal degree in $\mathbb{P}^{k+1}$. Since $2\mathcal{G}$ is a subsystem of codimension at most 2 in $\mathcal{H}$, we are in case (1).

If $\iota = 1$ then either $d = k$ or $k = 3$ and $d \geq 3$. In the latter case we are in case (2). In the former case if $h_Y(2) = 4k + 3$, then the curve section of $Y$ is linearly normal of degree $k$ and arithmetic genus 1 (see Theorem 1.1) and we are in case (3). If $h_Y(2) = 4k + 4$ then, again by Theorem 1.1, the curve section of $Y$ cannot be linearly normal of degree $k$ and arithmetic genus 1, hence it has arithmetic genus 0 and it is therefore smooth. By Lemma 1.3 we are in case (4).

If $\iota = 2$, then all the inequalities in (9) are equalities, hence $X$ is the 2–uple embedding of $Y$. We have either $k = 4$ and $d \geq 5$ or $d = k + 1$ and $k \geq 4$. If $k = 4$, the equalities in (9) imply that $Y$ is contained in a quadric of $\mathbb{P}^5$ and we are in case (5). If $d = k + 1$, then $Y$ has degree $k + 1$. Since equalities hold in (9), by Theorem 1.1, a general curve section of $Y$ is linearly normal of degree $k + 1$, with arithmetic genus 2 and we are in case (6). ∎

EXAMPLE 4.3. *We show that if $X$ is as in any one of the previous cases, then it is minimally $k$–defective. We may assume $k \geq 2$.*

(1) Assume that $X \subset \mathbb{P}^r$ is contained in a cone $W$ over the 2-uple embedding $Y' \subset \mathbb{P}^{4k+1}$ of a threefold $Y$ of minimal degree in $\mathbb{P}^{k+1}$ with vertex $L$ a line or a point. Accordingly one has $4k + 3 \geq r \geq 4k + 2$. Remark that by definition $X$ projects onto $Y'$ from $L$.

In Example 6.2 below it is proved that $Y'$ is $k$–defective and $s^{(k)}(Y') = 4k$. By Proposition 1.12, part (i), one has $s^{(k)}(X) \leq s^{(k)}(Y') + \dim(L) + 1 = 4k + \dim(L) + 1 < \sigma^{(k)}(X)$, thus $X$ is $k$–defective.

Let us prove that $X$ is minimally $k$–defective. In Example 6.2 below we will see that $Y'$ is minimally $(k-1)$–defective and that $s^{(k-1)}(Y') = 4k - 2$. Moreover the general $(k-1)$–contact locus of $Y'$ is an irreducible curve. Suppose, by contradiction, that $X$ is $(k-1)$–defective. Then by Proposition 1.12, part (i), we have $4k - 2 = s^{(k-1)}(Y') \leq s^{(k-1)}(X) < \sigma^{(k-1)}(X) = 4k - 1$, thus $s^{(k-1)}(X) = 4k - 2$. By Proposition 1.12, part (iii), $X$ it would be, as well as $Y'$, minimally $(k-1)$–defective. Moreover the general $(k - 1)$–contact locus of $X$ projects from $L$ to the general $(k - 1)$–contact locus of $Y'$, hence it is an irreducible curve. However we will see at the beginning of §6 below that the maximum for the embedding dimension $r$ of a minimally $(k - 1)$–defective threefolds whose general $(k - 1)$–contact locus is an irreducible curve is $r \leq 4k + 1$. This gives a contradiction, which proves that $X$ is not $(k - 1)$–defective.

If $L$ is a point, then $r = 4k + 2$ and by Proposition 1.15, part (v), we have $\delta_k(X) = 1$, $X_k$ is a plane curve and $s^{(k)}(X) = 4k + 1$.

If $L$ is a line, then $r = 4k + 3$. By Proposition 3.6, part (i), we have $n_k = 2$. Then Proposition 1.15, part (iii), yields $\delta_k(X) = 1$ hence $s^{(k)}(X) = 4k + 2$. Actually we see that in this case $X_k$ is a cone in $\mathbb{P}^3$. Indeed, by projecting from a general point of $L$ one has to go back to the previous situation.

(2) Let $Y$ be a threefold in $\mathbb{P}^4$ of degree $d \geq 3$. Let $X := Y'$ be the 2–uple embedding of $Y$ in $\mathbb{P}^{14}$. If $p$ is a point of $Y$ we abuse notation and denote by $p$ also the corresponding point on $Y'$.

We have $\sigma^{(3)}(X) = 14$. Let $p_0, ... p_3$ general points on $Y$. Since there is a quadric in $\mathbb{P}^4$ singular at $p_0, ... p_3$, namely the hyperplane $< p_0, ... p_3 >$ counted twice, then $T_{Y', p_0, ..., p_3}$ is contained in a hyperplane. This proves that $Y'$ is 3–defective.

Let us prove that $Y'$ is minimally 3–defective. Certainly $Y'$ is not 1–defective. Indeed let $p_0, p_1$ be general points on $Y$. There are hyperplane sections of $Y'$ which

have isolated singularities at $p_0, p_1$, namely the hyperplane sections of $Y$ corresponding to the quadrics in $\mathbb{P}^4$ singular along the line $< p_0, p_1 >$. Suppose that $Y'$ is 2–defective and therefore minimally 2–defective. Let $p_0, p_1, p_2$ be general points on $Y$. There are hyperplane sections of $Y'$ tangent at $p_0, p_1, p_2$ and having an irreducible singular locus of dimension 1, namely the hyperplane sections of $Y$ corresponding to the quadrics in $\mathbb{P}^4$ singular along the plane $< p_0, p_1, p_2 >$. Then, as we will seee at the beginning of §6 below, the embedding dimension of $Y'$ should be bounded above by 13, a contradiction.

By Proposition 1.15, part (v), we have $\delta_3(Y') = 1$, $s^{(3)}(Y') = 13$, $n_3(Y') = 1$ and $Y'_3$ is a plane curve.

Assume now that $X$ is contained in a cone with vertex a point $v$ over $Y'$, so that $r = 15$ and that $X$ maps to $Y'$ from $v$. By Proposition 1.12, part (i), we have $s^{(3)}(X) \leq s^{(3)}(Y') + 1 = 14$, hence $X$ is 3–defective. By the same Proposition 1.12, part (ii), since $Y'$ is minimally 3–defective, also $X$ is minimally 3–defective.

By Proposition 3.6, part (i), we have $n_k = 2$. Then Proposition 1.15, part (iii), yields $\delta_k(X) = 1$ hence $s^{(k)}(X) = 14$.

(3) Assume that $X$ is contained in a cone with vertex a point (or the empty set) over the 2-uple embedding $Y'$ of a threefold $Y$ of degree $k$ in $\mathbb{P}^{k+1}$ with linearly normal curve section of arithmetic genus 1. Then $Y'$ sits in $\mathbb{P}^{4k+2}$ (see Theorem 1.1). In [6], Example 4.7, it is proved that $Y'$ is $k$–defective and $s^{(k)}(Y') \leq 4k + 1$. An argument completely similar to the one of the previous Example (2) shows that $Y'$ is minimally $k$–defective.

If $X = Y'$ then Proposition 1.15, part (v), yields $\delta_k(X) = 1$, $n_k(X) = 1$, $s^{(k)}(Y') = 4k + 1$.

Assume $X \neq Y'$, hence $X \subset \mathbb{P}^{4k+3}$. Then Proposition 1.12, part (i), yields $s^{(k)}(X) \leq s^{(k)}(Y') + 1 = 4k + 2$, thus $X$ is $k$–defective. Again Proposition 1.12 implies it is minimally $k$–defective. Then by Proposition 3.6, part (i), we have $n_k = 2$. Finally Proposition 1.15, part (iii), yields $\delta_k(X) = 1$, hence $s^{(k)}(X) = 4k + 2$.

(4) There are threefolds $Y$ of degree $k$ in $\mathbb{P}^{k+1}$ with smooth curve sections of genus 0 and a singular line. They are described in Proposition 1.4. By Lemma 1.3, the 2–uple embedding $X$ of $Y$ sits in $\mathbb{P}^{4k+3}$. By arguing as in Example 4.5 or 4.7 of [6], one sees that $X$ is $k$–defective. Then an argument completely similar to the one of the previous Example (2) shows that $X$ is minimally $k$–defective. As usual we find $\delta_k(X) = 1$, $n_k = 2$ and $s^{(k)}(X) = 4k + 2$.

(5) Let $X \subset \mathbb{P}^{19}$ be the 2-uple embedding of a threefold $Y$ contained in a unique quadric of $\mathbb{P}^5$. If $p$ is a point of $Y$ we abuse notation and denote by $p$ also the corresponding point on $X$.

One has $\sigma^{(4)}(X) = 19$. However consider five general points $p_0, \ldots, p_4$ of $Y$. Since there is a quadric singular at any five points of $\mathbb{P}^5$, i.e. the double hyperplane $< p_0, ..., p_4 >$, then there is a hyperplane section of $X$ tangent at $p_0, ..., p_4$, hence $X$ is 4–defective. The usual kind of arguments show that $X$ is minimally 4–defective and that $\delta_4(X) = 1$, $n_4 = 2$ and $s^{(k)}(X) = 18$.

(6) Let $X$ be the 2–uple embedding of a threefold $Y$ of degree $k+1$ in $\mathbb{P}^{k+1}$ with curve sections of arithmetic genus 2. Then $X$ sits in $\mathbb{P}^{4k+3}$ by Thoerem 1.1. Moreover $X$ is $k$–defective. This follows by arguing as in [6], Example 4.7 in the usual way. Again one has $\delta_k(X) = 1$, $n_k = 2$ and $s^{(k)}(X) = 4k + 2$.

Notice that for all the above examples $X \subset \mathbb{P}^r$, one has $s^{(k)}(X) = r - 1$. Then, by Proposition 1.11, $s^{(k+1)}(X) = r$, i.e. $X$ is not $h$–defective for any $h \neq k$.

It is worth adding a few words about the existence of smooth threefolds of the above types. It is obvious that this can happen for threefolds of types (1), (2) and (5). Threefolds of types (3) and (6) can be smooth, but only for finitely many values of $k$ (this follows, for instance, from the results in [20]; more specifically, for the genus 1 case, see [23] for the genus 1 case). Threefolds of type (4) can never be smooth.

Next let us turn to the case $n = \dim(Y) = 2$. First we prove a lemma on the dimension of the system $2\mathcal{G}$.

LEMMA 4.4. *In the above setting, if $n = 2$ then $\dim(2\mathcal{G}) \leq r - 2$. If the equality holds, then $\Delta = 0$.*

*Proof.* Notice that $X$ is not a cone because it is $k$–mimimally defective, with $k > 1$.
 The system $2\mathcal{G}$ maps $X$ to a surface $Y$, then it cannot coincide with $\mathcal{H}$. Since $\mathcal{H}$ has no fixed parts, then $\dim(2\mathcal{G}) < r$.
 If $\dim(2\mathcal{G}) = r - 1$, then, as in Proposition 4.2, we see that $X$ is contained in the cone, with vertex a point, over $Y$, which is a surface. Then $X$ would be the cone over $Y$, a contradiction.
 If $\dim(2\mathcal{G}) = r - 2$, then $X$ sits in a cone over $Y$ with vertex a line $\ell$. The divisor $\Delta$ has to be contained in $\ell \cap X$, hence $\Delta = 0$. ∎

Now we are ready for the proof of the classification theorem in the case $n = 2$:

THEOREM 4.5. *In the above setting, assume $n = 2$. Then $r = 4k + 3 - i$, $i = 0, 1$ and one of the following cases occurs:*

  (1) *$X$ is contained in a cone with vertex a space of dimension $k - i$ over the 2–uple embedding of a surface $Y$ of minimal degree in $\mathbb{P}^{k+1}$;*
  (2) *$k = 2$, $i = 0$ and $X$ sits in a cone with vertex a line over the 2–uple embedding of a surface $Y$ of $\mathbb{P}^3$ with $\deg(Y) \geq 3$;*
  (3) *$k \geq 3$, $i = 0$ and $X$ sits in a cone with vertex of dimension $k - 1$ over the 2–uple embedding of a surface $Y$ of degree $k + 1$ in $\mathbb{P}^{k+1}$ with curve sections of arithmetic genus 1.*

*All threefolds in the above list are minimally $k$–defective with $\delta_k(X) = 1$, $s^{(k)}(X) = r - 1 = 4k + 3 - i$, $n_k = 2 - i$, and are not $h$–defective for any $h > k$.*

*Proof.* Recall that $r = 4k + 3 - i$ with $i = 0, 1$. Formula (9) and Lemma 4.4 give:

$$4k + 2 - i \geq (r + 1) - 2 = r - 1 \geq h_Y(2) \geq 3k + 3 + \iota$$

where $\iota = \min(d - k, k - 1)$. Then, by writing $h_Y(2) = 3k + 3 + q$, we see that $X$ is contained in a cone $W$ with vertex of dimension $s = r - 3k - q - 3 = k - q - i$ over the 2-uple embedding $Y'$ of a non degenerate surface $Y \subset \mathbb{P}^{k+1}$, such that $h_Y(2) = 3k + 3 + q$. By Proposition 1.18, $X$ is certainly $(s + 1)$–defective. By the minimality assumption we must have $k \leq s + 1 \leq k - q - i + 1$, thus $q + i \leq 1$.

If $q = 0$, by Theorem 1.1, $Y$ is a surface of minimal degree in $\mathbb{P}^{k+1}$. In this case $Y'$ is minimally $k$–defective (see Theorem (1.3) of [6]) and, Proposition 1.18 implies that $X$ is minimally $k$–defective too. One has $s^{(k)}(X) = s^{(k)}(Y') + k - i + 1 = 4k + 2 - i$, hence $\delta_k(X) = 1$ and $n_k(X) = 2 - i$.

If $q = 1$, then $i = 0$. Taking into account Theorem 1.1, we see that only the following cases may occur:

- $k = 2$ and $Y$ is a surface in $\mathbb{P}^3$ of degree $d \geq 3$ and we are in case (2);
- $k \geq 3$ and $Y$ is a surface of degree $k + 1$ in $\mathbb{P}^{k+1}$ with curve sections of arithmetic genus 1, and we are in case (3).

In both the above cases $Y$ is not $k$–defective (see Theorem (1.3) of [6]), and therefore by, Proposition 1.18, $X$ is minimally $k$–defective.                    ∎

REMARK 4.6. (1) Observe that in the last statement, in cases (1) and (3) we cannot specify whether $\Delta \equiv H - 2\Sigma$ is zero or not. When it is not, then $X$ meets the vertex of the cone along a divisor. In case (2) instead, $\Delta = 0$ by Lemma 4.4.

(2) A few words about the existence of smooth threefolds in the list of Theorem 4.5. For example, consider case (1). Give an embedding whatsoever of a smooth $Y$ surface of minimal degree in $\mathbb{P}^{k+1}$ in a $\mathbb{P}^{k-i}$. This is certainly possible if $k - i > 4$. Call $Y'$ the surface we get in this way. Then join any point of $Y'$ with the corresponding point of $Y$. The resulting ruled threefold is smooth.

With the same idea one produces smooth threefolds of type (3) in Theorem 4.5 as soon as $k \geq 6$. However note here that such a construction works only for $k \leq 8$, since, as it is well known, there are no smooth surfaces of degree $k + 1$ in $\mathbb{P}^{k+1}$ with curve sections of arithmetic genus 1 as soon as $k \geq 9$.

## 5. THE REDUCIBLE CASE

Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, minimally $k$–defective threefold with $k \geq 2$. In this section we examine the case in which the general element of the family $\mathcal{G}$ is reducible. By Proposition 3.1, we know that in this case $\Gamma = \Gamma_{p_0,\ldots,p_k}$ has exactly $k + 1$ components, one passing through each of the points $p_i$, $i = 0, \ldots, k$.

The following proposition immediately follows by Proposition 3.3, Remark 1.25:

PROPOSITION 5.1. *Let $X \subset \mathbb{P}^r$, be an irreducible, non–degenerate, projective, minimally $k$–defective threefold with $k \geq 2$. Suppose that the general tangential $k$–contact locus of $X$ is reducible. One has:*

(1) *if $\gamma_k = 1$, then $r \geq 4k + 3$, $\delta_k = 1$, $n_k = 2$ and the general $(k-1)$–tangential projection $X_{k-1}$ of $X$ is a cone over a surface which is not $1$–weakly defective;*

(2) *if $\gamma_k = 2$ and $n_k = 2$, then $r \geq 4k + 3$, $\delta_k = 1$ and the general $(k-1)$–tangential projection $X_{k-1}$ of $X$ is a threefold contained in a 4-dimensional cone over a curve;*

(3) *if $\gamma_k = 2$ and $n_k = 1$ then $r \geq 4k + 2$ and the general $(k-1)$–tangential projection $X_{k-1}$ of $X$ is a cone with vertex a line over a curve.*

*Conversely, threefolds like in case (1), (2) and (3) above are minimally $k$–defective with reducible general tangential $k$–contact locus.*

We can now specify case (1) of Proposition 5.1:

PROPOSITION 5.2. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate projective threefold. Then the following are equivalent:*

(i) *the general $(k-1)$–tangential projection $X_{k-1}$ of $X$ is a cone over a surface which is not 1–weakly defective;*

(ii) *$X$ sits in a cone of dimension $k+2$, and not smaller, over a surface which is not $k$–weakly defective.*

*Proof.* Supppose the general $(k-1)$–tangential projection $X_{k-1}$ of $X$ is a cone over a non–developable surface $S$, hence $X$ is minimally $k$–defective. Part (i) is trivially true for $k = 1$. So we assume $k \geq 2$ and proceed by induction on $k$.

Let $p_0 \in X$ be a general point and let $\tau := \tau_{X,1} : X \dashrightarrow X_1$ be the tangential projection from $T_{X,p_0}$. By Proposition 5.1 and Proposition 3.3, $\gamma_{k-1}(X_1) = 1$ and the general tangential $(k-1)$–contact locus of $X_1$ is a reducible curve.

By induction, we know that $X_1$ is contained in a cone over a surface $S_1$ with vertex $\Pi := \Pi_{p_0}$ of dimension $k - 2$. Notice that the general tangent space to $X_1$ meets $\Pi$ at one point and projects from $\Pi$ to the general tangent space to $S_1$ (see Proposition 1.17).

Call $\pi : X \dashrightarrow S_1$ the composition of $\tau$ with the projection of $X_1$ from $\Pi$. For a general point $p \in X$, call $Z_p$ the intersection of $X_1$ with $< \Pi, \pi(p) >$, which is the general fibre of the projection of $X_1$ from $\Pi$. Hence $Z_p$ is a curve.

Take general points $p_1, \ldots, p_k \in X$. Terracini's lemma and Proposition 1.18 imply that $T_{X_1, \tau(p_1), \ldots, \tau(p_k)} = < \Pi, T_{S, \pi(p_1), \ldots, \pi(p_k)} >$, hence $T_{X_1, \tau(p_1), \ldots, \tau(p_k)}$ is tangent to $X_1$ along the curves $Z_{p_i}$, $i = 1, \ldots, k$. Furthermore $T_{X_1, \tau(p_1), \ldots, \tau(p_k)}$ is tangent to $X_1$ along the pull–back of the general tangential $(k-1)$–contact locus to $S_1$. Since, as we saw, the general tangential $(k-1)$–contact locus of $X_1$ is a reducible curve, we have that $S_1$ is not $(k-1)$–weakly defective.

Notice that, if we move $p_1$ to a new general point $p_1'$, then $T_{X_1, \tau(p_1'), \ldots, \tau(p_k)} = < \Pi, T_{S, \pi(p_1'), \ldots, \pi(p_k)} >$ is again tangent to $X_1$ along the curves $Z_{p_i}$, $i = 2, \ldots, k$.

Call $C_i$ the pull–back to $X$ of $Z_{p_i}$ via $\tau$. One has that $T_{X, p_0, p_1, \ldots, p_k}$ is tangent to $X$ along a curve which contains all the $C_i$'s. In other words $\Gamma_{p_0, p_1, \ldots, p_k} \supset C_1 \cup \cdots \cup C_k$. By what we saw above, if we move the point $p_1$ to a new general point $p_1'$, then $T_{p_0, p_1', p_2 \ldots, p_k}$ is again tangent along the curves $C_2, \ldots, C_k$. Therefore also by moving $p_0$ to some other point $p_0' \in X$, again $\Gamma_{p_0', p_1, \ldots, p_k}$ contains the curves $C_i$, $i = 1, \ldots, k$.

By equation (5), we know that $h(\Gamma) \leq 2(1+k) - 1 = 2k + 1$, so $\dim(< \Gamma >) \leq 2k$. On the other hand we claim that:

$$(10) \qquad\qquad \dim(< \Gamma >) \geq k - 1 + \dim(< C_i >)$$

Otherwise there is an $h < k$ such that $\Lambda := < C_1, \ldots, C_h >$ contains all curves $C_i$, with $h < i \leq k$. Then, by the genericity of the points $p_1, \ldots, p_k$, the space $\Lambda$ would contain the whole of $X$, but this is impossible, since:

$$\dim(< \Lambda >) \leq \dim(< \Gamma >) \leq 2k < 4k + 2 \leq r.$$

By (10) it follows that $\dim(< C_i >) \leq k + 1$.

Set $L := L_{p_0} = < T_{X,p_0}, \Pi >$. By Proposition 1.18, $X$ cannot lie on a cone on a surface with vertex $\Pi$. Thus $\dim(L) = k + 2$. We have:

$$\dim(< Z_{p_i} >) = \dim(< C_i >) - \dim(< C_i > \cap T_{X,p_0}) - 1$$
$$\dim(< Z_{p_i} > \cap \Pi) = \dim(< C_i > \cap L) - \dim(< C_i > \cap T_{X,p_0}) - 1.$$

Since $\dim(< Z_{p_i} > \cap \Pi) = \dim(< Z_{p_i} >) - 1$ because $Z_{p_i}$ projects from $\Pi$ to a point of $S$, we have that also $H_i := < C_i > \cap L$ is a hyperplane in $< C_i >$, $i = 1, \ldots, k$.

Now move the point $p_0$ to a new point $p_0' \in X$, which gives us a new vertex $\Pi' := \Pi_{p_0'}$ and a new space $L' := L_{p_0'} = < T_{X,p_0'}, \Pi' >$. However, since $\Gamma_{p_0', p_1, \ldots, p_k}$ contains the curves $C_i$, $i = 1, \ldots, k$, we have that also $< C_i > \cap L'$ is a hyperplane in $< C_i >$, $i = 1, \ldots, k$.

Now $\dim(< L, L' >) \leq 2k + 5 < 4k + 2 \leq r$, so that $C_i$ cannot be contained in $< L, L' >$, for a general choice of $p_i$, otherwise $X$ would be contained in $< L, L' >$. Hence the hyperplane $H_i$ of $< C_i >$ is fixed, as $p_0$ moves. Then the space $< H_1, \ldots, H_k >$ does not depend on $p_0$, i.e. it is contained in $L_p$ for a general point $p \in X$.

Consider now the linear space $H = \cap_{P \in U} L_P$, where $U$ is a suitable dense open subset of $X$. Notice that $L_p$ has to vary when $p$ varies in $X$, because $T_{X,p} \subset L_p$. Hence $\dim(H) \leq \dim(L_p) - 1 = k + 1$. Moreover $H$ contains all the hyperplanes $H_i$ of $< C_i >$ so it meets the tangent line of $C_i$ at $P_i$, which is a general point of $X$. Thus $H$ intersects all the tangent spaces to $X$. We conclude by Proposition 1.17 that $X$ projects from $H$ either to a curve or to a surface.

If $\dim(H) \leq k - 1$, then by Proposition 1.18 and since $X$ is minimally $k$–defective, we have that $\dim(H) = k - 1$ and $X$ projects from $H$ to a surface $S$ which is not $k$–weakly defective, proving (ii).

Assume $\dim(H) \geq k$. Then since for a general point $p \in X$ we have $H \cup T_{X,p} \subset L_p$ and $\dim(L_p) = k + 2$ then $H$ would meet $T_{X,p}$. If $\dim(H) = k$ then $\dim(H \cap T_{X,p}) \geq 1$ and, by Proposition 1.17, we would have that $X$ sits in a cone with vertex $H$ over a curve. Then, by applying Proposition 1.18, we see that $X$ would be $(k-1)$–defective, a contradiction. If $\dim(H) = k + 1$ then $\dim(H \cap T_{X,P_0}) \geq 2$ and, by Proposition 1.17, $X$ would be contained in a subspace of dimension $\dim(H) + 1 = k + 2$ a contradiction.

We have thus completed the proof of the fact that (i) implies (ii).

Let us assume now that (ii) holds. Then, by Proposition 1.18, $X$ is minimally $k$–defective and, since $S$ is not $k$–weakly defective, then $\gamma_k(X) = 1$ and the general tangential $k$–contact locus is a reducible curve. Then, by Proposition 5.1, (i) holds. ∎

The following proposition is proved with a similar argument, but with a slight modification:

PROPOSITION 5.3. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate projective three-fold. Let $k \geq 2$ be an integer. Then the following are equivalent:*
  (i) *the general $(k-1)$–tangential projection $X_{k-1}$ of $X$ sits in a 4–dimensional cone over a curve;*
  (ii) *$X$ sits in a cone of dimension $2k + 2$, and not smaller, over a curve.*

*Proof.* . Let us prove that (i) implies (ii). The assertion trivially holds for $k = 2$. So we assume $k \geq 3$ and proceed by induction on $k$.

Let $p_0 \in X$ be a general point and let $\tau := \tau_1 : X \dashrightarrow X_1$ be the tangential projection from $T_{X,p_0}$. By induction, we know that $X_1$ is contained in a cone over a curve $C$ with vertex $\Pi := \Pi_{p_0}$ of dimension $2k - 2$. Call $\pi : X \dashrightarrow C$ the composition of $\tau$ with the projection of $X_1$ from $\Pi$. For a general point $p \in X$, call $Z_p$ the intersection of $X_1$ with the span $< \Pi, \tau(p) >$, which is the fibre of the projection of $X_1$ from $\Pi$. Hence $Z_Q$ is a surface.

Take general points $p_1, \ldots, p_k \in X$. One has:

$$\dim(T_{C,\pi(p_1),\ldots,\pi(p_k)}) \leq 2k - 1$$

hence:

$$\dim(< \Pi, T_{C,\pi(p_1),\ldots,\pi(p_k)} >) \leq 4k - 2$$

and $T_{X_1,\tau(p_1),\ldots,\tau(p_k)} \subseteq < \Pi, T_{C,\pi(p_1),\ldots,\pi(p_k)} >$, so that:

$$\dim(T_{X_1,\tau(P_1),\ldots,\tau(P_k)}) \leq 4k - 2.$$

Since $\delta_k(X) = 1$ (see 1stclass ) the equality has to hold in the previous inequality, which implies $T_{X_1,\tau(p_1),\ldots,\tau(p_k)} = < \Pi, T_{C,\pi(p_1),\ldots,\pi(p_k)} >$. Furthermore $T_{X_1,\tau(P_1),\ldots,\tau(P_k)}$ is tangent to $X_1$ along the surface $Z_{P_i}$, $i = 1, \ldots, k$.

Call $S_i$ the pull–back to $X$ of $Z_{P_i}$ via $\tau$. It follows that $T_{X,p_0,p_1,\ldots,p_k}$ is tangent to $X$ along a surface which contains all the $S_i$'s. In other words $\Gamma_{p_0,p_1,\ldots,p_k} \supset S_1, \ldots, S_k$. In particular the tangential $k$–contact locus of $X$ is reducible. As in the proof of Proposition 5.2 we see that by moving $p_0$ to some other point $p_0' \in X$, then $\Gamma_{p_0',p_1,\ldots,p_k}$ also contains the surfaces $S_i$, $i = 1, \ldots, k$.

By equation (5), one has $\dim(< \Gamma >) \leq 3k + 1$. On the other hand, as in the proof of proposition 5.2, one has $\dim(< \Gamma >) \geq k - 1 + \dim(< S_i >)$. Hence $\dim(< S_i >) \leq 2k + 2$.

Set $L := L_{P_0} = < T_{X,p_0}, \Pi >$, which is a linear space of dimension $2k + 2$. Again one sees that $H_i := < S_i > \cap L$ is a hyperplane in $< S_i >$, $i = 1, \ldots, k$.

Now move the point $p_0$ to a new point $p_0' \in X$, which gives us a new vertex $\Pi' := \Pi_{p_0'}$ and a new space $L' := L_{p_0'} =< T_{X,P_0'}, \Pi' >$. However, since $\Gamma_{p_0',p_1,...,p_k}$ contains $S_i$, $i = 1, \ldots, k$, we have that also $< S_i > \cap L'$ is a hyperplane in $< S_i >$, $i = 1, \ldots, k$.

Now we claim that $\dim(< L, L' >) < r$. In fact, $< T_{X,p_0}, T_{X,p_0'} >$ contains $T_{X_1,\tau(p_0')}$ and $\Pi$ intersects this space along a line. Hence $\dim(< T_{X,p_0}, T_{X,p_0'}, \Pi >) \leq 2k + 4$. For analogous reasons, $\Pi'$ intersects $< T_{X,p_0}, T_{X,p_0'} >$ at least along a line. Hence $\dim(< L, L' >) \leq 4k + 1 < r$.

Since $< L, L' >$ is a proper subspace of $\mathbb{P}^r$, the general surface $S_i$ cannot be contained in it. Hence the hyperplane $H_i$ of $< S_i >$ is fixed, as $p_0$ moves. Then the space $< H_1, \ldots, H_k >$ does not depend on $p_0$, i.e. it is contained in $L_p$ for all points $p \in X$.

Now one concludes exactly as in the proof of Proposition 5.2.

Let us prove now that (ii) implies (i). Suppose $X$ lies in a cone with vertex $\Pi$ of dimension $2k$ over a curve $C$. By Proposition 1.19, $X$ is minimally $k$–defective, hence $X_{k-1}$ is a threefold. Since the general tangent space to $X$ meets $\Pi$ along a line, in the $(k-1)$–tangential $\Pi$ projects to a line $L$, and $X_{k-1}$ sits in the cone with vertex $L$ over $C_{k-1}$.

∎

With similar arguments, using Proposition 1.19, one proves the following result:

PROPOSITION 5.4. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate projective three-fold. Then the following are equivalent:*

(i) *the general $(k-1)$–tangential projection $X_{k-1}$ of $X$ is a cone over a curve;*
(ii) *$X$ sits in a cone of dimension $k+2$ over a $k$–defective surface with reducible general tangential $k$–contact locus, hence $X$ sits in a cone of dimension $2k+1$, and not smaller, over a curve.*

The previous results finish the classification in this case.

REMARK 5.5. (1) The minimally $k$–defective threefolds occurring in Propositions 5.2, 5.3 and 5.4 can be $h$–defective for some $h > k$.

To be more precise, consider a threefold as in Proposition 5.2 and let $h > k$. We know that $X \subset \mathbb{P}^r$ lies on a cone with vertex a subspace $\Pi$ of dimension $k - 1$ over a surface $S$. By applying Proposition 1.18, we see that $s^{(h)}(X) = s^{(h)}(Y) + k$. For example, if $S$ is not $h$–defective, then $s^{(h)}(X) = \min\{r, 3h + k + 2\}$, so that $X$ is $h$–defective if and only if $3h + k + 2 < r$. If $S$ is defective, $X$ is defective even for higher values of $h$. We leave the details to the reader.

Similarly, if $X$ is as in Proposition 5.3 [resp. Proposition 5.4] a similar argument shows that, for any $h > k$, one has $s^{(h)}(X) = \min\{r, 2h + 2k + 2\}$ [resp. $s^{(h)}(X) = \min\{r, 2h + 2k + 1\}$], so that $X$ is $h$–defective if and only if $2h + 2k + 2 < r$ [resp. $2h + 2k + 1 < r$].

(2) A threefold as in Proposition 5.2 can be smooth. This can be proved with a construction analogous to the one proposed in Remark 4.6, part (2). Similarly one

can prove the existence of smooth threefolds as in Proposition 5.3. Take indeed a smooth scroll surface $Y$ over a curve $C$ spanning a $\mathbb{P}^{2k}$. Embed $C$ in a $\mathbb{P}^{r-2k-1}$. Then join every point of $C$ with the corrresponding line of $Y$. The resulting scroll threefold $X$ is smooth. An analogous construction works for producing smooth threefolds as in Proposition 5.4.

## 6. THE IRREDUCIBLE CURVILINEAR CASE

Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, minimally $k$–defective threefold with $k \geq 2$. In this section we examine the case $\gamma_k(X) = 1$ and the general element of the family $\mathcal{G}$ is irreducible.

By Proposition 3.3 we know that the general tangential projection $X_{k-1}$ is a 1–defective threefold whose general contact locus is an irreducible curve. From Remarkm 1.25 it follows that $X_{k-1}$ as in cases (4) or (5) of Theorem 1.24. In any event, we have that $X_{k-1}$ sits in $\mathbb{P}^s$, $s = 7, 8, 9$, hence:

$$r \in \{4k + 3, 4k + 4, 4k + 5\}.$$

Now we can start our classification. The first step it to show that $X$ is as described in Proposition 1.26.

PROPOSITION 6.1. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, minimally $k$–defective threefold with $k \geq 1$. Assume $\gamma_k = 1$ and the general element $\Gamma$ of the family $\mathcal{G}$ irreducible. Then $\Gamma$ is a rational normal curve in $\mathbb{P}^{2k}$. Hence $X$ is rationally connected and therefore* regular, *i.e. any desingularization $X'$ of $X$ has $h^1(X', \mathcal{O}_{X'}) = 0$.*

*Proof.* The assertion holds for $k = 1$ (see Remark 1.25). So we may assume $k > 1$ and proceed by induction on $k$.

Let $p \in X$ be a general point, let $\tau := \tau_1 : X \dashrightarrow X_1$ be the tangential projection from $T_{X,p}$ and let $\Gamma_1$ be the general tangential $(k-1)$–contact locus of $X_1$. By induction, $\Gamma_1$ is a rational normal curve in $\mathbb{P}^{2k-2}$ and $\Gamma$ maps onto $\Gamma_1$ via $\tau$ (see the proof of Proposition 3.3).

One has $\dim(< \Gamma >) = 2k$. Indeed by (5) we have $h(\Gamma) \leq 2k + 1$, moreover the center of the projection $\tau$, that is the tangent space $T_{X,p}$, meets $< \Gamma >$ at least in the tangent line to $\Gamma$ at $p$ and the image of $\Gamma$ via $\tau$ spans a $\mathbb{P}^{2k-2}$. It follows also that, in particular, $< \Gamma >$ meets $T_{X,p}$ exactly along the aforementioned tangent line, so that $\tau_{|\Gamma}$ is a general tangential projection of $\Gamma$.

Now we claim that:
  (i) $\Gamma$ is not tangentially degenerate;
  (ii) if $x \in \Gamma$ is a general point, the projection $\Gamma'$ of $\Gamma$ from $x$ is also not tangentially degenerate.

The assertions in the statement about $\Gamma$ follow from (i) and (ii). Indeed, by (ii) and Lemma 1.23, $\tau_{|\Gamma}$ is birational to its image $\Gamma_1$, hence $\Gamma$ is rational. Furthemore, since $\deg(\Gamma_1) = 2k - 2$ by induction, $< \Gamma >$ meets $T_{X,p}$ exactly along $T_{C,p}$, as we saw, and $T_{C,p} \cap C = \{p\}$ by (i), we have $\deg(\Gamma) = 2k$.

To prove (i) and (ii), we notice that $\mathcal{G}$ is a family of curves of dimension $2(k+1)$. This is an easy count of parameters wich follows from the fact that there is a unique curve of $\mathcal{G}$ containing $k+1$ general points of $X$. Let $p_1, \ldots, p_k$ be general points of $X$ and let $\mathcal{G}'$ be the 2–dimensional family of curves of $\mathcal{G}$ passing through $p_1, \ldots, p_k$. We claim that:

(iii) the tangent lines to the curves of $\mathcal{G}'$ at $P_i$ fill up the whole tangent space $T_{X,p_i}$, for all $i = 1, \ldots, k$.

Indeed, this is true for $k = 1$ (see Remark 1.25). Then proceed by induction. The curves in $\mathcal{G}'$ are mapped via $\tau_1 = \tau_{X,p_1}$ to the contact curves on $X_1$ through $\tau_1(p_i)$, $i = 2, \ldots, k$. Since the differential of $\tau_1$ is an isomorphism at $p_2, \ldots, p_k$, which are general points, by induction the tangent lines to the curves of $\mathcal{G}'$ at $p_i$ fill up the tangent space $T_{X,p_i}$, for all $i = 2, \ldots, k$. Arguing by monodromy (see §monodromy and the proof of Proposition 3.1) the same is true for $i = 1$.

Now we claim that (iii) implies (i). Indeed, if $\Gamma$ were tangentially degenerate, then, by (iii), $X$ itself would be 2-tangentially degenerate. By Proposition 1.22, $X$, which does not lie in $\mathbb{P}^4$, would be a scroll on a curve. Then also $X_{k-1}$ would be a scroll, which is not the case (see Remark 1.25).

Furthermore (iii) also implies (ii). In fact if we apply the same argument as above to the projection $X'$ of $X$ from a general point $x \in X$, we conclude that $X'$ would be a scroll. Let $\Pi$ be a general plane of $X'$. Then either $\Pi$ pulls back to a plane on $X$ or it pulls back to a quadric $Q$ containg $x$. In the former case $X$ would be a scroll, which, as we saw, is impossible. In the latter case $X$ is swept out by a family $\mathcal{Q}$ of quadrics such that through two general points of $X$ there is a quadric in $\mathcal{Q}$ containing them. In this case $X$ would be 1-defective (see Proposition 1.26), which is against the minimality assumption. ∎

To go on with the classification, we need the following:

EXAMPLE 6.2. Let $Y \subset \mathbb{P}^{k+2}$ be a threefold of minimal degree $k$ and let $X \subset \mathbb{P}^{4k+5}$ be its double embedding. In [6], Example 4.5, it is proved that $X$ is $(k+1)$–defective and $s^{(k+1)}(X) \leq 4k + 4$. One can prove that $X$ is actually minimally $k$–defective, that $s^{(k)}(X) = 4k+2$ and $s^{(k+1)}(X) = 4k+4$, so that $\delta_k(X) = 1$, $n_k = 1$. Actually we will see that $\gamma_k(X) = 1$ and the general tangential $k$–contact locus is an irreducible curve.

First we observe that $X$ is not $h$–weakly defective, hence not $h$–defective, for any $h \leq k - 1$. Indeed, if $p_0, \ldots, p_h \in X$ are general points, there are hyperplane sections of $X$ tangent at $p_0, \ldots, p_h$ and having isolated singularities at $p_0, \ldots, p_h$. Take, for instance the hyperplane sections corresponding to sections of $Y$ with a general quadric cone with vertex along the span of the points corresponding to $p_0, \ldots, p_h$. By the way, the same argument applied for for $h = k$, shows that there are hyyerplane sections of $X$ tangent at $p_0, \ldots, p_k$ and having an irreducible curve of singular points containing $p_0, \ldots, p_k$, i.e. the rational normal curve of degree $2k$ which is the immage of the intersection of $Y$ with the span of the points corresponding to $p_0, \ldots, p_k$.

Now suppose $X$ is not $k$–defective. Then, if $p_0, \ldots, p_k \in X$ are general points, we would have $\dim(T_{X,p_0,\ldots,p_k}) = 4k + 3$. However, if $p \in X$ is another general point,

one has $\dim(T_{X,p_0,\dots,p_k,p}) \leq 4k + 4$. This would imply that the general tangent space $T_{X,p}$ to $X$ intersects $T_{X,p_0,\dots,p_k}$ in dimension at least 2. But this would force $X$ to span a $\mathbb{P}^{4k+4}$ (see Proposition 1.17), a contradiction. Hence $X$ is minimally $k$–defective and therefore $s^{(k)}(X) = 4k + 2$, $\delta_k(X) = 1$, $n_k = 1$ (see Proposition 1.15). Then the same argument implies that $s^{(k+1)}(X) \geq 4k + 4$ and therefore $s^{(k+1)}(X) = 4k + 4$.

Of course also a projection of $X$ to $\mathbb{P}^{4k+3+i}$, $i = 0, 1$, is minimally $k$–defective.

To apply an inductive argument we will need the following:

PROPOSITION 6.3. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, regular threefold. Let $k \geq 2$ and assume that a general tangential projection $X_1$ of $X$ is the 2-uple embedding of a minimal threefold in $\mathbb{P}^{k+1}$. Then $X$ is the 2-uple embedding of a minimal threefold in $\mathbb{P}^{k+2}$.*

*Proof.* Let $\tau := \tau_1 : X \dashrightarrow X_1$ be the general tangential projection from $T_{X,p}$.

Let $Z$ [resp. $Z_1$] be a desingularization of $X$ [resp. $X_1$] and let $\mathcal{H}$ [resp. $\mathcal{H}_1$] be the pull–back on $Z$ [resp. $Z_1$] of the hyperplane linear system of $X$ [resp. $X_1$]. The tangential projection $\tau$ induces a rational map $\phi : Z \dashrightarrow Z_1$. Notice that:

$$(11) \qquad \phi^*(\mathcal{H}_1) = \mathcal{H}(-2p).$$

By the hypothesis $X_1$ spans a $\mathbb{P}^{4k+1}$, so $r = 4k + 5$. Furthermore $X_1$ is linearly normal, namely $\mathcal{H}_1$ is a complete linear system. By (11) also $\mathcal{H}$ is complete, i.e. $X$ is also linearly normal.

By assumption, $\mathcal{H}_1$ is the double of a linear system $\mathcal{L}_1$ and the associated map $\phi_{\mathcal{L}_1} : Z_1 \dashrightarrow Y_1 \subset \mathbb{P}^{k+1}$ is such that $Y_1$ is a threefold of minimal degree. The linear system $\mathcal{L}_1$ pulls back via $\phi$ to a linear system $\mathcal{L}_p$ on $Z$. Of course $\phi_{\mathcal{L}_p} = \phi_{\mathcal{L}_1} \circ \phi : Z \dashrightarrow Y_1 \subset \mathbb{P}^{k+1}$ and therefore $\dim(\mathcal{L}_p) = k + 1$. The system $\mathcal{L}_p$ depends on $p$, but, since $X$ is regular, as $p$ moves on $X$ all the systems $\mathcal{L}_p$'s are subsystems of the same complete linear system $\mathcal{L}$ on $Z$.

We notice that:

$$(12) \qquad 2\mathcal{L}_p = \phi^*(\mathcal{H}_1) = \mathcal{H}(-2p).$$

Hence all divisors in $\mathcal{L}_p$ contain $p$ and therefore we have

$$(13) \qquad \mathcal{L}_p \subseteq \mathcal{L}(-p)$$

As an imediate consequence of (12) and (13) we have the linear equivalence relation $2\mathcal{L} \equiv \mathcal{H}$. Since $\mathcal{H}$ is complete, we have $2\mathcal{L} \subseteq \mathcal{H}$.

Moreover, by (13), one has $m := \dim(\mathcal{L}) \geq k + 2$. We call $Y \subset \mathbb{P}^m$ the image of $Z$ via the map $\phi_{\mathcal{L}}$ associated to $\mathcal{L}$.

If $m > k+2$, then $\dim(\mathcal{H}) \geq \dim(2\mathcal{L}) > 4k+5$ (see Theorem 1.1), a contradiction. Similarly we find a contradiction if $\dim(\mathcal{H}) > \dim(2\mathcal{L})$. Hence $\mathcal{H} = 2\mathcal{L}$ and $m = k+2$. Moreover for $p$ general, $\mathcal{L}_p = \mathcal{L}(-p)$. It follows that $X$ is the 2-uple embedding

of $Y$ and $Y \subset \mathbb{P}^{k+2}$ projects from a general point on it to a minimal threefold of $\mathbb{P}^{k+1}$. Hence $Y$ is also of minimal degree. ∎

REMARK 6.4. *In the previous setting, assume that a general tangential projection $X_1$ of $X$ is a projection of the 2-uple embedding $Y'$ of a threefold $Y$ of minimal degree in $\mathbb{P}^{k+1}$, from a space not intersecting $Y'$. Then $X$ is the projection of the 2-uple embedding of a threefold of minimal degree in $\mathbb{P}^{k+2}$, from a space not intersecting it.*

Indeed, by replacing, if necessary, $X_1$ by its linearly normal embedding, the previous argument shows that the linearly normal embedding of $X$ is the 2–uple embedding of a minimal threefold in $\mathbb{P}^{k+2}$.

A completely similar argument holds when $X_1$ is a projection of the 2-uple embedding of a minimal threefold from a point on it. We skip the details.

PROPOSITION 6.5. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, regular threefold. Let $k \geq 2$ and assume that a general tangential projection $X_1$ of $X$ is the projection of the 2-uple embedding $Y'$ of a threefold $Y$ of minimal degree in $\mathbb{P}^{k+1}$ from a space which meets $Y'$ at a point. Then $X$ is the projection of the 2–uple embedding of a threefold of minimal degree in $\mathbb{P}^{k+2}$ from a space which meets it at a point.*

The case of projections from more than one point on $X$ is contained in the analysis below. Again, in order to apply an inductive argument, we will need the following:

PROPOSITION 6.6. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, regular threefold. Let $k \geq 2$ and assume that a general tangential projection $X_1$ of $X$ is linearly normal and contained in the Segre embedding of $\mathbb{P}^k \times \mathbb{P}^k$. Suppose that each of the two projections of $X_1$ to $\mathbb{P}^k$ spans $\mathbb{P}^k$. Then $X$ is linearly normal and contained in the Segre embedding of $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$. Moreover each of the two projections of $X$ spans $\mathbb{P}^{k+1}$.*

*Proof.* Let $\tau := \tau_1 : X \dashrightarrow X_1$ be the general tangential projection from $T_{X,P}$.

Let $Z$ [resp. $Z_1$] be a desingularization of $X$ [resp. $X_1$] and let $\mathcal{H}$ [resp. $\mathcal{H}_1$] be the pull–back on $Z$ [resp. $Z_1$] of the hyperplane linear system of $X$ [resp. $X_1$]. The tangential projection $\tau$ induces a rational map $\phi : Z \dashrightarrow Z_1$. Equation (11) still holds. Then, as in the proof of Proposition 6.3, one sees that $X$ is linearly normal, i.e. $\mathcal{H}$ is complete.

We have two linear systems $\mathcal{A}_1, \mathcal{B}_1$ on $Z_1$ which come by pulling back the linear system $|\mathcal{O}_{\mathbb{P}^k}(1)|$ via the projections of $\mathbb{P}^k \times \mathbb{P}^k$ to the two factors. One has $\mathcal{H}_1 = \mathcal{A}_1 + \mathcal{B}_1$ (see §1.4.

The linear systems $\mathcal{A}_1, \mathcal{B}_1$ pull–back via $\phi$ to linear systems $\mathcal{A}_p, \mathcal{B}_p$, on $Z$. By the regularity assumption on $X$, as $p$ varies $\mathcal{A}_p, \mathcal{B}_p$ vary inside two complete linear systems $\mathcal{A}, \mathcal{B}$ on $Z$. From the relation:

$$(14) \qquad \mathcal{A}_p + \mathcal{B}_p = \phi^*(\mathcal{H}_1) = \mathcal{H}(-2p)$$

we see that every divisor in $\mathcal{A}_p + \mathcal{B}_p$ vanishes doubly at $p$, and therefore either:

$$(15) \qquad \qquad \mathcal{A}_p \subseteq \mathcal{A}(-p), \mathcal{B}_p \subseteq \mathcal{B}(-p)$$

or, say:

$$(16) \qquad \qquad \mathcal{B}_p \subseteq \mathcal{B}(-2p)$$

In any case we have the equivalence relation $\mathcal{A} + \mathcal{B} \equiv \mathcal{H}$. Since $\mathcal{H}$ is complete, then

$$(17) \qquad \qquad \mathcal{A} + \mathcal{B} \subseteq \mathcal{H}.$$

In case (15) holds, we have:

$$(18) \qquad \qquad \mathcal{A}_p + \mathcal{B}_p \subseteq \mathcal{A}(-p) + \mathcal{B}(-p) \subseteq \mathcal{H}(-2p)$$

Then, by (14), equality has to hold everywhere in (18) and therefore in (15) and (17). This yields $\dim(\mathcal{A}) = \dim(\mathcal{B}) = k + 1$, proving the assertion.

If (16) holds, one has:

$$(19) \qquad \qquad \mathcal{A}_p + \mathcal{B}_p \subseteq \mathcal{A} + \mathcal{B}(-2p) \subseteq \mathcal{H}(-2p)$$

Again, by (14), equality has to hold everywhere in (19). From (17) we also have $\mathcal{A}(-p) + \mathcal{B}(-p) \subseteq \mathcal{H}(-2p)$ which is clearly incompatible with $\mathcal{H}(-2p) = \mathcal{A} + \mathcal{B}(-2p)$.
∎

We can now get the following partial classification result:

THEOREM 6.7. *Let $X \subset \mathbb{P}^r$ be an irreducible, non–degenerate, projective, minimally $k$–defective threefold with $k \geq 2$. Assume $\gamma_k = 1$ and irreducible general tangential $k$–contact locus. Then one of the following holds:*

(1) *$r = 4k + 5$ and $X$ is the 2-uple embedding of a threefold of minimal degree in $\mathbb{P}^{k+2}$;*

(2) *$r = 4k + 4$ and $X$ is the projection of the 2-uple embedding of a threefold of minimal degree in $\mathbb{P}^{k+2}$ from a point $p \in \mathbb{P}^{4k+5}$;*

(3) *$r = 4k + 3$ and either $X$ is the projection of the 2-uple embedding of a threefold of minimal degree in $\mathbb{P}^{k+2}$ from a line $\ell \subset \mathbb{P}^{4k+5}$, or $X$ is linearly normal, it is contained in the intersection of a space of dimension $4k + 3$ with the Segre embedding of $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$ in $\mathbb{P}^{k^2+4k+3}$ and each of the two projections of $X$ on the two factors of $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$ spans $\mathbb{P}^{k+1}$. In this former case each of the linear systems $\mathcal{A}_i$, $i = 1, 2$, on $X$ corresponding to the two projections $\phi_i : X \to \mathbb{P}^{k+1}$, is base point free and the general surface $A_i \in \mathcal{A}_i$ is irreducible.*

*Proof.* We know that the possibilities for $r$ are the ones listed in statement.

Assume that $r = 4k + 5$. Then, by Theorem 1.24, $X_{k-1}$ is the 2-uple embedding of $\mathbb{P}^3$. The conclusion follows by induction from Proposition 6.3.

If $r = 4k + 4$, by Theorem 1.24, $X_{k-1}$ is a projection of the 2-uple embedding of $\mathbb{P}^3$. The conclusion follows by induction from Remark 6.4 and Proposition 6.5.

If $r = 4k + 3$, then by Proposition 3.3, by Theorem 1.24 and Remark 1.25, either $X_{k-1}$ is some projection of the 2-uple embedding of $\mathbb{P}^3$ or $X_{k-1}$ is a hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2$. In the former case the conclusion follows again by induction from Remark 6.4 and Proposition 6.5. In the latter case, $X_{k-1}$ is linearly normal and the two projections of $X_{k-1}$ to $\mathbb{P}^2$ are both surjective. The conclusion follows by induction from Proposition 6.6.                                                                              ∎

The last part of case (3) of Theorem 6.7 deserves more attention. The analysis is based on the following proposition:

PROPOSITION 6.8. *Let $Z$ be an irreducible, smooth, regular threefold and let $\mathcal{A}_i$, $i = 1, 2$, be two distinct base point free linear systems of dimension $k + 1 \geq 3$ on $Z$ such that a general surface $A_i \in \mathcal{A}_i$ is smooth and irreducible and the minimal sum $\mathcal{A}_1 + \mathcal{A}_2$ is birational on $Z$. Then the minimal sum $\mathcal{A}_1 + \mathcal{A}_2$ has dimension at least $4k + 3$.*

*Furthermore if the minimum is attained then a general curve $C$ in the class $A_1 \cdot A_2$ is smooth, irreducible and rational and the linear system $\mathcal{A}_{i|C}$ is the complete $g_k^k$ on $C$. Moreover, either:*

  (a) *$\mathcal{A}_i$, $i = 1, 2$, is complete, or*
  (b) *$A_1 \equiv A_2$, and $\dim(|A_i|) = k + 2$, $i = 1, 2$, or*
  (c) *$\mathcal{A}_1$ is complete, whereas $\dim(|A_2|) = k + 2$ and there is an effective, non–zero divisor $E$ on $X$, such that $h^0(Z, \mathcal{O}_Z(E)) = 1$, $A_2 - A_1 \equiv E$ and $\mathcal{O}_{A_2}(E) = \mathcal{O}_{A_2}$.*

*If, in addition, the linear systems $\mathcal{A}_i$, $i = 1, 2$, are very big, then also the general curves $C_i$ in the classes $A_i^2$ are smooth, irreducible and rational. Moreover:*

  (a') *in case (a) above the linear system $\mathcal{A}_{i|C_i}$ is the complete $g_{k-1}^{k-1}$ on $C_i$, $i = 1, 2$;*
  (b') *in case (b) above the linear system $\mathcal{A}_{i|C_i}$ is the complete $g_k^k$ on $C_i$, $i = 1, 2$;*
  (c') *in case (c) above the linear system $\mathcal{A}_{1|C_1}$ is the complete $g_{k-1}^{k-1}$ on $C_1$ whereaes the linear system $\mathcal{A}_{2|C_2}$ is the complete $g_k^k$ on $C_2$.*

*Proof.* The curve $C$ is smooth by Bertini's theorem. We claim that $C$ is also irreducible. Otherwise, by Bertini's theorem, the linear system $\mathcal{A}_{1|A_2}$ would be composite with a pencil $\mathcal{P}$ and, if $P \in \mathcal{P}$ is a general curve, we would have $P \cdot A_1 = 0$. Similarly we would have $P \cdot A_2 = 0$ and therefore each curve $P$ would be contracted by the map $\phi_{\mathcal{A}_1 + \mathcal{A}_2}$, against the hypothesis that $\mathcal{A}_1 + \mathcal{A}_2$ is birational.

Consider the linear systems $\mathcal{A}_{i|C}$, $i = 1, 2$. Since $\mathcal{A}_1 \neq \mathcal{A}_2$, one has:

$$\text{(20)} \qquad\qquad \dim(\mathcal{A}_{i|A_j}) = k + 1, i, j = 1, 2, i \neq j$$

and therefore:

$$\text{(21)} \qquad\qquad \dim(\mathcal{A}_{i|C}) = k, i = 1, 2.$$

By Lemma 1.5 we have $\dim(\mathcal{A}_{1|C} + \mathcal{A}_{2|C}) \geq 2k$.

Let $\dim(\mathcal{A}_{1|A_1} + \mathcal{A}_{2|A_1}) = s$. By looking at the exact sequence:

$$0 \to H^0(Z, \mathcal{O}_Z(A_2)) \to H^0(Z, \mathcal{O}_Z(A_1 + A_2)) \to H^0(A_1, \mathcal{O}_{A_1}(A_1 + A_2))$$

we see that the image of the rightmost map contains the vector space corresponding to $\mathcal{A}_{1|A_1} + \mathcal{A}_{2|A_1}$. This implies that:

$$\dim(\mathcal{A}_1 + \mathcal{A}_2) \geq s + 1 + \dim(\mathcal{A}_2) = s + k + 2.$$

Similarly, by looking at the sequence:

$$0 \to H^0(A_1, \mathcal{O}_{A_1}(A_1)) \to H^0(A_1, \mathcal{O}_{A_1}(A_1 + A_2)) \to H^0(C, \mathcal{O}_C(A_1 + A_2))$$

one obtains:

$$s + 1 \geq \dim(\mathcal{A}_{1|C} + \mathcal{A}_{2|C}) + \dim(\mathcal{A}_{1|A_1}) + 2 \geq 3k + 2$$

whence $\dim(\mathcal{A}_1 + \mathcal{A}_2) \geq 4k + 3$ follows.

If the equality holds, then Lemma 1.5 implies that $C$ is rational. Since, by the hypotheses and by Lemma 1.8, the linear series $\mathcal{A}_{1|C} + \mathcal{A}_{2|C}$ is birational, then Lemma 1.5 again implies that $\mathcal{A}_{1|C} = \mathcal{A}_{2|C}$ and that this is a complete linear series. Since $C$ is rational, by (21) we have $A_i \cdot C = k$.

Since $\mathcal{A}_{1|C} = \mathcal{A}_{2|C}$ are complete, it follows that the systems $\mathcal{A}_{1|A_2}$ and $\mathcal{A}_{2|A_1}$ are also complete. If $h^0(Z, \mathcal{O}_Z(A_1 - A_2)) = h^0(Z, \mathcal{O}_Z(A_2 - A_1)) = 0$, then, by looking at the exact sequence:

(22)
$$0 \to H^0(Z, \mathcal{O}_Z(A_i - A_j)) \to H^0(Z, \mathcal{O}_Z(A_i)) \to H^0(A_j, \mathcal{O}_{A_j}(A_i)), i, j = 1, 2, i \neq j$$

we see we are in case (a).

Suppose we are not in case (a) and assume that $h^0(Z, \mathcal{O}_Z(A_2 - A_1)) > 0$, so that there is an effective divisor $E$ which is linearly equivalent to $A_2 - A_1$. Suppose $E$ is zero. Then $A_1 \equiv A_2$. Moreover since $\mathcal{A}_{2|A_1}$ is complete, from the sequence (22) with $i = 2, j = 1$, we see that $h^0(Z, \mathcal{O}_Z(A_2)) = \dim(\mathcal{A}_{2|A_1}) + 2 = k + 3$, namely we are in case (b).

If $E$ is non zero, then $h^0(Z, \mathcal{O}_Z(A_1 - A_2)) = 0$. Since $\mathcal{A}_{1|A_2}$ is complete, the completeness of $\mathcal{A}_1$ follows from the exact sequence (22) with $i = 1, j = 2$. Moreover $E \cdot C = 0$. Since the linear system $|C|$ on $A_2$ is base point free, with positive self–intersection, we see that $h^0(A_2, \mathcal{O}_{A_2}(E)) = 1$. From the exact sequence:

$$0 \to H^0(Z, \mathcal{O}_Z(-A_1)) \to H^0(Z, \mathcal{O}_Z(E)) \to H^0(A_2, \mathcal{O}_{A_2}(E))$$

we deduce that $h^0(Z, O_Z(E)) = 1$. Now look at the sequence (22) with $i = 2, j = 1$. Since, as we saw, $h^0(A_1, \mathcal{O}_{A_1}(A_2)) = \dim(\mathcal{A}_{2|A_1}) + 1 = k + 2$, we have $h^0(Z, \mathcal{O}_Z(A_2)) = k + 3$, i.e. $\dim(|A_2|) = k + 2$. Now, inside $|A_2|$ we have the two linear systems $E + \mathcal{A}_1$ and $\mathcal{A}_2$. They intersect along a $k$–dimensional linear system. Hence every section of $H^0(Z, \mathcal{O}_Z(A_2))$ determining a divisor of $\mathcal{A}_2$ is of the form $fe + g$, with $f \in H^0(Z, \mathcal{O}_Z(A_1))$ variable, $e \in H^0(Z, \mathcal{O}_Z(E))$ and $g \in H^0(Z, \mathcal{O}_Z(A_2))$

fixed defining a divisor $B\mathcal{A}_2$ not containing $E$. Hence every solution of the system $e = g = 0$, i.e. any point in $E \cap B$, should be a base point of $\mathcal{A}_2$. Since $\mathcal{A}_2$ is base point free, we have that $E \cap B = \emptyset$. Thus we are in case (c).

Suppose now $\mathcal{A}_i$, is very big. Arguing as at the beginning of the proof, we see that the general curve $C_i$ in the class $A_i^2$ is smooth and irreducible. By what we proved already, we have the exact sequence:

$$0 \to \mathcal{O}_{A_1}(-A_1) \to \mathcal{O}_{A_1}(A_2 - A_1) \to \mathcal{O}_C(A_2 - A_1) \simeq \mathcal{O}_C \to 0$$

from which we deduce $h^0(A_1, O_{A_1}(A_2 - A_1)) = 1, h^1(A_1, O_{A_1}(A_2 - A_1)) = 0$. Indeed, by Lemma 1.8, $\mathcal{O}_{A_1}(A_1)$ is big and nef and therefore Kawamata-Viehweg vanishing theorem says that $h^i(A_1, \mathcal{O}_{A_1}(-A_1)) = 0$, $0 \le i \le 2$.

Now look at the sequence:

$$0 \to \mathcal{O}_{A_1}(A_2 - A_1) \to \mathcal{O}_{A_1}(C) \to \mathcal{O}_{C_1}(C) \to 0.$$

Since $\mathcal{A}_{2|A_1} \subset |\mathcal{O}_{A_1}(C)|$, by (20) we have $h^0(A_1, O_{A_1}(C)) \ge k + 2$. It follows that $h^0(C_1, O_{C_1}(C)) \ge k + 1$. Since $\deg(O_{C_1}(C)) = k$, we deduce that $C_1$ is rational.

Suppose we are in case (a). Since $Z$ is regular and $\mathcal{A}_1$ is complete, we have:

$$h^0(A_1, \mathcal{O}_{A_1}(C_1)) = h^0(A_1, \mathcal{O}_{A_1}(A_1)) = \dim(\mathcal{A}_1) = k + 1.$$

Finally look at the sequence:

$$0 \to \mathcal{O}_{A_1} \to \mathcal{O}_{A_1}(C_1) \to \mathcal{O}_{C_1}(C_1) \to 0.$$

Clearly $A_1$ is a rational surface, so we have $h^1(A_1, \mathcal{O}_{A_1}) = 0$. Thus one has $h^0(C_1, \mathcal{O}_{C_1}(C_1)) = k$. This proves the assertion for $\mathcal{A}_{1|C_1}$. The same for $C_2$. Thus we are in case (a').

The analysis in case (b) and (c) is the same, leading to (b') and (c') respectively. ∎

COROLLARY 6.9. *Let $X$ be an irreducible, non–degenerate threefold in the Segre embedding of $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$, $k \ge 2$, which does not lie in the 2–uple embedding of $\mathbb{P}^{k+1}$. Assume that each of the two projections of $X$ to $\mathbb{P}^{k+1}$ spans $\mathbb{P}^{k+1}$. Then $X$ spans a space of dimension at least $4k + 3$.*

*Furthermore, if $X$ spans a $\mathbb{P}^{4k+3}$, then given $k + 1$ general points of $X$, there is a rational normal curve $C$ of degree $2k$ on $X$ containing the given points, and $X$ is minimally $k$–defective and $s^{(k)}(X) = 4k + 2$, $\delta_k(X) = 1$, $n_k(X) = 2$.*

*Finally, if the two projections of $X$ to $\mathbb{P}^{k+1}$ are generically finite to their images, then either:*

(a) *they both map birationally $X$ to rational normal scrolls, and then the degree of $X$ is $8k - 2$, or*

(b) *they both map birationally $X$ to projections of rational normal scrolls in $\mathbb{P}^{k+2}$ from a point, and then the degree of $X$ is $8k$, or*

(c) *one of them maps birationally $X$ to a rational normal scroll and the other maps birationally $X$ to the projection of a rational normal scroll in $\mathbb{P}^{k+2}$ from a point, and then the degree of $X$ is $8k - 1$.*

*Proof.* The first assertion follows from Proposition 6.8 applied to a desingularization $Z$ of $X$. Consider the two linear systems $\mathcal{A}_i$, $i = 1, 2$, on $Z$ corresponding to the two projections $X \to \mathbb{P}^{k+1}$. Notice that the general surface of $\mathcal{A}_i$, $i = 1, 2$, is irreducible (see Proposition 6.7) and smooth by Bertini's theorem, since $\mathcal{A}_i$ is base point free.

Suppose $X$ spans a $\mathbb{P}^{4k+3}$. Given $k + 1$ general points $p_0, \ldots, p_k$ on $Z$, let $A_i \in \mathcal{A}_i$ be the surface containing $p_0, \ldots, p_k$. Then, according to Proposition 6.8, the image in $X$ of the curve $C = A_1 \cdot A_2$ is a rational normal curve of degree $2k$. The $k$–defectivity of $X$ follows from Proposition 1.26.

Let us prove that $X$ is minimally $k$–defective. We first claim that $X$ is not 1–defective. Assume, by contradiction, that $X$ is 1-defective. Then, by Theorem 1.24, the only possibilities are that $X$ is either a cone, or $X$ sits in a 4–dimensional cone over a curve. Notice that the Segre embedding of $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$ is swept out by two $(k + 1)$–dimensional families of $\mathbb{P}^{k+1}$'s and that each line on the Segre embedding of $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$ is contained in a $\mathbb{P}^{k+1}$ of either one of these two families. As a consequence, each irreducible cone contained in the Segre embedding of $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$ is contained in a $\mathbb{P}^{k+1}$. This implies that $X$ cannot be a cone, since it spans a $\mathbb{P}^{4k+3}$. Suppose $X$ sits in a 4–dimensional cone over a curve. Let $p \in X$ be a general point and consider the general tangential projection $\tau$ of $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$ from $p$. As we saw in Example 1.16, $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$ projects onto $\mathbb{P}^k \times \mathbb{P}^k$. Let $X'$ be the image of $X$ via $\tau$. Notice that $X'$ is a projection of the image $X_1$ of the tangential projection of $X$ from $T_{X,p}$. Thus $X'$ is a cone over a surface. By the above argument, $X'$ would span at most a $\mathbb{P}^k$. Thus $4k + 3 = \dim(< X >) = \dim(< X' >) + 2k + 3 \leq 3k + 3$, a contradiction.

Let us consider again a general point $p \in X$ and the general projection $\tau_p$ from $T_{X,p}$. Since $X$ is not 1–defective, its image is a threefold spanning a $\mathbb{P}^{4k-1}$. Consider again a desingularization $Z$ of $X$. We abuse notation and denote by $p$ the point of $Z$ corresponding to $p \in X$. Let $\mathcal{H}$ be the linear system on $Z$ corresponding to the hyperplane section system on $X$. We have $\mathcal{A}_1(-p) + \mathcal{A}_2(-p) \subseteq \mathcal{H}(-2p)$. On the other hand, since $\dim(\mathcal{A}_i(-p)) = k$, $i = 1, 2$, by Proposition 6.8, we have $4k - 1 = \dim(\mathcal{H}(-2p)) \geq \dim(\mathcal{A}_1(-p) + \mathcal{A}_2(-p)) \geq 4k - 1$. This proves that $\mathcal{A}_1(-p) + \mathcal{A}_2(-p) = \mathcal{H}(-2p)$. On the other hand $\phi_{\mathcal{H}(-2p)}$ just maps $Z$ to $X_1$. By what we saw in §1.4, we have that $X_1$ is a threefold which sits in $\mathbb{P}^k \times \mathbb{P}^k$, spans a $\mathbb{P}^{4k-1}$ and each of the two projections of $X_1$ to $\mathbb{P}^k$ spans $\mathbb{P}^k$. Thus, by arguing as above, we see that $X_1$ is not 1–defective. By iterating this argument, one proves that $X_i$ is not 1–defective for any $i = 1, ..., k - 2$, hence that $X$ is minimally $k$–defective.

Now, notice that the above argument proves that $X_{k-1}$ is the hyperplane section of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$, hence $s^{(1)}(X_{k-1}) = 6$, $\delta_1(X_{k-1}) = 1$, $n_1(X_{k-1}) = 2$. Thus $s^{(k)}(X) = 4k + 2$, $\delta_k(X) = 1$, $n_k(X) = 2$.

Finally, the cases (a), (b) and (c) in the statement, correspond to the homologous case in Proposition 6.8. ∎

The following proposition takes care of the cases left out in the statement of Corollary 6.9.

PROPOSITION 6.10. *Let $X$ be an irreducible, non–degenerate threefold in the Segre embedding of $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$, $k \geq 2$, which lies in the 2-uple embedding of $\mathbb{P}^{k+1}$. Suppose the two projections of $X$ to $\mathbb{P}^{k+1}$ span $\mathbb{P}^{k+1}$. Then these projections coincide and are immersions of $X$ in $\mathbb{P}^{k+1}$. Furthermore the dimension $s$ of the linear span of $X$ satisfies $s > 4k + 3$ unless:*

- (a) *$s = 4k + 1$ and the image of $X$ in $\mathbb{P}^{k+1}$ is a threefold $Y$ of minimal degree;*
- (b) *$s = 4k + 2$ and the image of $X$ in $\mathbb{P}^{k+1}$ is a threefold $Y$ of degree $k$ with curve sections of arithmetic genus 1;*
- (c) *$s = 4k + 3$ and the image of $X$ in $\mathbb{P}^{k+1}$ is:*
  - (c$_1$) *either a threefold $Y$ of degree $k + 1$ with curve sections of arithmetic genus 2 or*
  - (c$_2$) *a threefold $Y$ of degree $k$ which is the projection in $\mathbb{P}^{k+1}$ of a threefold of minimal degree in $\mathbb{P}^{k+2}$. In the latter case $Y$ is described in Lemma 1.3.*

*The threefolds in case (a) are $(k - 1)$–defective, whereas the threefolds in cases (b), (c) are minimally $k$–defective.*

*Proof.* In the present situation it is clear that the two projections of $X$ to $\mathbb{P}^{k+1}$ coincide and are immersions $\phi : X \to Y \subset \mathbb{P}^{k+1}$ of $X$ in $\mathbb{P}^{k+1}$. Let $d$ be the degree of $Y$. Set $d = k - 1 + \iota$. Now we make free and iterate use of Theorem 1.1. So we have $s \geq 4k + 1 + \iota$. If $\iota = 0$ we are in case (a). If $\iota = 1$ and $s = 4k + 2$ we are in case (b). If $\iota = 1$ and $s = 4k + 3$, the curve sections of $Y$ cannot have arithmetic genus 1, otherwise $s = 4k + 2$. Then the curve sections of $Y$ have arithmetic genus 0, and we are in case $(c_2)$, as described in Lemma 1.3. Finally if $\iota = 2$ and $s = 4k + 3$ we are in case $(c_1)$. In all other cases $s \geq 4k + 4$.

The defectivity of the varieties in the list follows by Example 6.2, (6), Example 4.3 (3), (4), (6). ∎

EXAMPLE 6.11. (1) It is clear that there are examples of smooth threefolds as in cases (1) and (2) of Theorem 6.7. It is also not difficult to find examples of smooth threefolds as in case (3) of the samme theorem, and specifically as in cases (a), (b), (c) of Corollary 6.9.

For instance, take a threefold $Y$ of minimal degree in $\mathbb{P}^{k+2}$ and call $\mathcal{H}$ its hyperplane linear system. Put $\mathcal{A}_i = \mathcal{H}(-p_i)$, $i = 1, 2$, with $p_1, p_2 \in \mathbb{P}^{k+2}$ distinct points. Then $\mathcal{A}_1 + \mathcal{A}_2$ embeds $Y$ in the product $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$ and also in $\mathbb{P}^{4k+3}$, since it corresponds to a projection of the 2–uple embedding of $Y$ from the line $\ell = < p_1, p_2 >$. Notice that these projections have degree $d$ with $d = 8k, 8k - 1, 8k - 2$, according to the fact that $\ell$ does not intersect $Y$, meets $Y$ at one point, or it is a secant of $Y$.

(2) There are examples of minimally $k$-defective threefolds, contained in the intersection of the Segrre embedding of $\mathbb{P}^{k+1} \times \mathbb{P}^{k+1}$ with a $\mathbb{P}^{4k+3}$, for which one the linear systems $\mathcal{A}_i$ of Proposition 6.8 is not very big.

This is the case of the threefold $X$ obtained by embedding $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^{19}$ with the complete system $\mathcal{A}$ of divisors of type $(1,3)$. Terracini pointed out in [31] that $X$ is 4–defective. Terracini's paper was reconsidered, from a modern point of view, by Dionisi and Fontanari ([12]), who also proved that the $X$ is not 3–defective. Let us show how this threefold fits in our classification.

The linear system $\mathcal{A}$ can be decomposed as the sum $\mathcal{A}_1 + \mathcal{A}_2$ with $\mathcal{A}_1$ of type $(1,1)$, $\mathcal{A}_2$ of type $(0,2)$. Both $\mathcal{A}_1$, $\mathcal{A}_2$ send $\mathbb{P}^1 \times \mathbb{P}^2$ to $\mathbb{P}^5$, so $X$ also sits in the product $\mathbb{P}^5 \times \mathbb{P}^5$, and the projections to the two factors are both non–degenerate. Therefore $X$ fits in the last case (15) of our classification. According to Proposition 6.8, a general curve in the class $\mathcal{A}_1 \cdot \mathcal{A}_2$ is rational: indeed it is embedded as a rational normal quartic in the natural Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$.

Observe, however, that $\mathcal{A}_2$ is not big: the second projection of $\mathbb{P}^5 \times \mathbb{P}^5$ sends $X$ to a surface. So we cannot apply to $X$ the last part of Proposition 6.8 as well as the last part of Corollary 6.9. In particular the degree of $X$ is $27 < 30 = 8k - 2$ and $X$ is not the projection of the 2–uple embedding of a quartic threefold in $\mathbb{P}^6$ from a line.

Other similar examples of this sort can be considered and are related to the so–called phenomenon of *Grassmann defectivity* (see [12]). We hope to come back on this subject in a future paper.

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